

# Development of the Work Energy Theorem in Mechanics



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**Introduction.** The concepts of work and energy are introduced in all beginning physics classes. At that beginning level, the student has attained a proficiency in algebra and trigonometry only, and as such, many educators believe the student is unable to appreciate or understand the development of the work energy theorem. These educators typically present work as force times displacement, kinetic energy and potential energy as magic formulas, and spend much time discussing how these energies transform from one form to the next. Problems are authored and solved as if all problems can be solved by “conservation of energy” and that it, energy, is *always* conserved. Textbook sections entitled “Optional - Work done by a variable force.”, clearly set the tone of *lack of importance* of this situation, for students and instructors alike. Some text book authors do address the idea that work is an integral part of the energy concept, but fail to paint the big picture of path independence or dependence, which ties all the work energy components together in a logical fashion, favoring to create a number of “special cases” that the student memorizes, but rarely understands. The fact that *it is not necessary to know how the energy transforms from form to form when solving problems*, the real power of the work energy theorem, is totally missed.

This condemnation of physics education may in fact seem somewhat harsh. Textbook authors could use the afore noted lack of math skill as argument to defend their approach to the work energy omission. Still, there is an understandable thread connecting work and energy, path function, point function, and that story needs to be told because it provides the foundation for understanding (or lack thereof) in thermodynamics courses to follow.

In areas of physics, history will help convey a sense of the concept. With force, we have but to look to Newton to see how this concept was developed. Newton’s laws are tightly packaged in a brief moment in history. Work energy, however, has no moment in history and no founding father. Work energy is similar to a meal in reverse, the dessert served first, and lastly the meat and potatoes. The work energy story cannot be told in chronological order because it was not developed in any logical order. It simply evolved.

The goal of this paper is to present the work energy theorem to the student of calculus based physics in an order that is logical, not historical. To show the beauty of the arrangement, the power of the results, so that for the student, the Work Energy Theorem finally falls together with crystal clarity.

**Prolog, In search of *Vis Viva*.** Prior to the time of Galileo (1546-1642 A.D.), science was conducted largely by philosophical argument. Ideas were purposed, and elegant arguments proposed to lend reason to the premise. In the Aristotelian world view, experimentation was perceived to be “unnatural” and therefore not appropriate for discerning the workings of the natural world.

It is Galileo who is thought of as the father of experiment. His cleverness in designing experiments, which verify premise, provided a new way of knowing, and experiment is considered the standard for the verification of premise today.

At Galileo's time in history, the intellects of the day were in search of the **cause** variable, the *Vis Viva*, the “living force” of motion. Experiments involving the collision of balls were conducted. Huygens (1629-1695) and others would notice that if a moving ball were to collide into a string of stationary balls of the same mass, that the moving ball would stop and a ball at the other end of the string would start motion, having the same velocity as which the incident ball previously had. This experiment and others of similar nature presented a peculiar outcome. An outcome, which seemed to suggest that something of an unusual nature, had occurred. It was thought that if this experiment could be understood, one might discover the *Vis Viva*, the cause for the effect.

We now understand this experiment in terms of both momentum and energy. Toys, consisting of swinging balls able to collide, are commonplace. Yet, in the days of Galileo, the experimental outcome could only be considered peculiar. Humanity was in no position to fully understand the principles involved and could only classify the result. Enter the era of Newton (1642-1727).

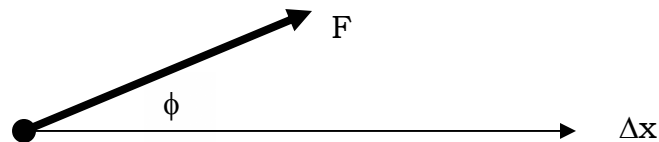
Around 1687 A.D., Newton publishes his thoughts of Force. Newton's three force laws provide a new understanding of force, acceleration, impulse, and momentum. The conceptual tools needed to understand all of mechanics are in place. The Calculus is also invented by Newton, providing a mathematical tool to manipulate these physical laws. Newton provides the tools that will ultimately open the pathway to the understanding of what is now called the work energy theorem. Newton does not complete the concept however and many other scientists will contribute to its evolution.

By the time of Helmholtz (1821-1894), and with the help of Bernoulli, Lagrange, Laplace, and others, the work energy theorem would be understood. The stage would be set for Carnot (1796-1832) to extend the work energy theorem to its next logical level, Thermodynamics! Clasius, Kelvin, Maxwell, Boltzman, Nernst, and finally Gibbs, would evolve thermodynamics to its full glory. We now tell the story in a much different order.

**Work.** Work is where we start our logical discussion. The concept of work can be interpreted as the measurement of the accomplishment of a task. To do work, effort (force) is required. Having effort however does not mean that work is done. The effort must accomplish something...a displacement must accompany the effort. If either the force or the displacement is zero, the work will be zero. Another requirement is that the force must be in the direction of the displacement. If the force is not in the direction of the displacement, only the component of the force, in the direction of the displacement, can be used to compute the work. For the special case where the force is constant, the work may be computed as:

$$Work = |\vec{F}||\Delta\vec{x}|\cos\phi,$$

where  $\phi$  is the angle between the force and displacement vector when arranged tail-to-tail. The above work equation is often written in shorthand, dot product, notation as:  $Work = \vec{F} \bullet \Delta\vec{x}$



When using the word work, one should keep in mind the following sentence: **Work done by \_\_\_\_\_, on \_\_\_\_\_**. The first blank is filled in by the force in question. The remaining blank specifies the object that displaces. Examples:

Work done by my force on the ball.  
 Work done by the force of gravity on the ball.  
 Work done by the net force on the ball.

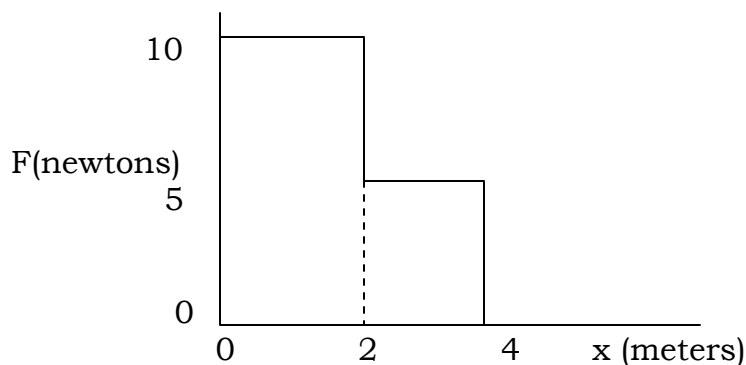
The target force, or force group, should always be specified.

Work is a scalar quantity. The sign on the calculation is controlled by the cosine of the angle. If the angle is less than 90 degrees, the work is positive. If the angle is greater than 90 degrees, the work is negative. If the angle is 90 degrees, the work will be zero.

Since work is a scalar, the total work done for a sequence of tasks is just the algebraic sum of the work done for each task in the sequence.

The formula,  $Work = |\vec{F}||\Delta\vec{x}|\cos\phi$ , will compute work as long as the *force is constant*, however for a variable force, a more general expression,

an integral, must be used. Consider the computation of the work done by the variable force in the figure below:



Our tactic will be to view the problem as a sequence of works. First, the work done to displace the initial two meters, and then the work done to displace the final two meters.

- We have a constant force of 10 newtons with a displacement of 2 meters. The work done is  $Work = |10 \text{ n}| |2\text{m}| \cos 0 = 20 \text{ nm}$  (or joules).
- We have a constant force of 5 newtons with a displacement of 2 meters. The work done is  $Work = |5\text{n}| |2\text{m}| \cos 0 = 10 \text{ joules}$ .

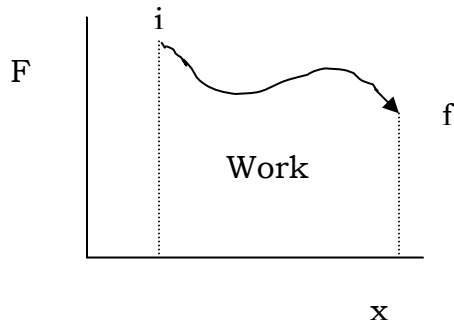
The total work done is thus  $20\text{j} + 10\text{j} = 30 \text{ joules}$ . Notice that in the above computations, we have effectively computed the area of two rectangles! Our total work is the area under the force vs displacement graph. This is true in all situations. We have only to devise a clever way of computing this area and we can compute the work done for any variable force.

The clever computation method, called an integral, is in effect an “*exact approximation*”. The integral in calculus is developed by first cutting the area under the curve into rectangular strips of equal width ( $\Delta x$ ). The approximate area is then computed by summing the area of all strips. The exact area is then obtained by looking at the trend when the strip width,  $\Delta x$ , is made infinitely small and the number of strips is made correspondingly infinitely large.

$$Work = \sum_1^{\infty} F_i \cdot \Delta x, \text{ where } \Delta x \rightarrow 0 \equiv \int F \cdot dx$$

here  $dx$  is considered an infinitely small  $\Delta x$ . This is the definition of the calculus integral.

By using a graph of force verses displacement, even the non-calculus physics student can understand work done by a variable force. Areas can be computed from polygon formulas and/or approximated using rectangular strips.



When computing work in this manner, the student is cautioned to double check the sign on the resulting area by inspection of the force and displacement vectors involved. Indeed, work viewed as area under the force displacement graph is the *perfect* application to introduce the student to the concept of an integral. It is also the perfect description of a Path Function. In moving from the point “i” to the point “f”, it is easily understood that the area under the curve depends on the path chosen to connect points “i” and “f”.

Work is defined as:

$$Work \equiv \int dw \equiv \int \vec{F} \cdot d\vec{s}$$

When applying the integral form, one should first think of expressing  $dw$ , the small quantity of work along a limiting short part of the path, then, add up all the small work increments to arrive at the total work.

Work is in general, a **PATH** function. The term path is used since one must know how the force varies all along the displacement path. The path connecting the start and end point determines the amount of work.

**Work done by the Net Force to change an objects Speed.** Here we explore the work calculation targeting the sum of the vector forces ( $F_{net}$ ) on a single point mass (or center of mass). Starting with the definition of work,

$$Work_{F_{net}} \equiv \int \vec{F}_{Net} \cdot d\vec{s}$$

Since from Newton,  $F_{net} = m a$ , we substitute for  $F_{net}$ ,

$$Work_{F_{net}} = \int (m\vec{a}) \cdot d\vec{s}$$

By definition,  $a = dv/dt$  and so we substitute for  $a$ ,

$$Work_{F_{net}} = \int \left( m \frac{d\vec{v}}{dt} \right) \cdot d\vec{s}$$

We now remove the mass from the integral as CONSTANT,

$$Work_{F_{net}} = m \int \frac{d\vec{v}}{dt} \cdot d\vec{s}$$

We re-associate  $dt$  with  $ds$ ,

$$Work_{F_{net}} = m \int d\vec{v} \cdot \frac{d\vec{s}}{dt}$$

By definition,  $ds/dt$  is velocity, we substitute.

$$Work_{F_{net}} = m \int d\vec{v} \cdot \vec{v}$$

We now evaluate the dot product by reasoning that  $dv$  could be broken into two components, one in the direction of  $v$ , and one component at 90 degrees to  $v$ . The cosine applied to the 90 degree component produces zero (a direction change but not a speed change). This leaves us with,

$$Work_{F_{net}} = m \int_{v_i}^{v_f} v dv = \frac{1}{2} m (v_f^2 - v_i^2) = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

This outcome is much unexpected; Where in general work is a path function, we find that the work done by the net force is NOT a path function at all. It is a **POINT** function. The work done by the net force is determined by the initial and final speeds alone, not by any speeds along the path, nor by the directions of any of these speeds. This, *is Discovery!* So much so, that we will create a new quantity, Kinetic Energy =  $\frac{1}{2} m v^2$ , for which the work done by the net force is the change in kinetic energy.

The word energy, coming from the German language (Gr. *energōs*, active; from *en*, in + *ergon*, work), is best translated as *the ability to do work, by virtue of having some unique condition or privilege of position*. Energy is not work. It is the ability to do work. As we have seen from the above derivation, work is done when energy changes. The change in energy is work. “Kinetic” identifies the unique condition – *motion*.

We now re-invent and rename the work done by the net force; and introduce the fundamental version of the Work Energy Theorem.

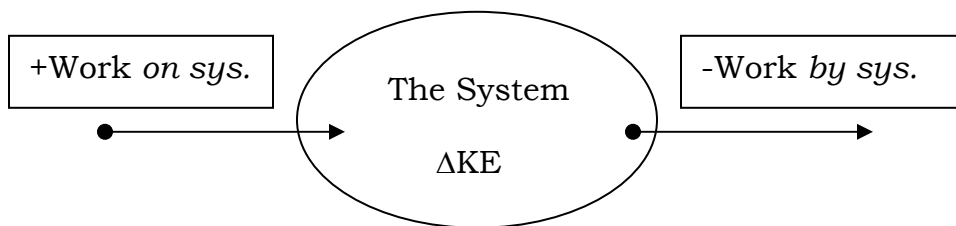
$$Work_{By F_{net}} = \Delta KE$$

where

$$\Delta KE \equiv \frac{1}{2} m (v_f^2 - v_i^2)$$

This outcome is the fundamental step in the development of other refined versions of the work energy theorem. From experiment, to theory. It was waiting for the tools of Newton.

As currently formulated, we now have the basis of an *Accounting System* and the sign convention also has its basis in accounting, not direction. When positive work is done, the energy of the system increases. When negative work is done, the energy of the system decreases.



Work by  $F_{net}$ , changes the Kinetic energy of the System.

**Interlude. Looking at the Big Picture.** One part of this picture we have not yet seen is that the work done by the net force, is equal to the sum of the individual works, done by all of the individual forces that comprise the net force. This fact is easy to show by derivation and example. Let  $F_G$ ,  $F_E$ ,  $F_f$ , represent gravitational, elastic, frictional, etc, general force categories.

$$\begin{aligned} Work_{By F_{net}} &= \int \vec{F}_{net} \cdot d\vec{x} = \int (\vec{F}_G + \vec{F}_E + \vec{F}_{Ff} + \dots) \cdot d\vec{x} = \int (\vec{F}_G \cdot d\vec{x} + \vec{F}_E \cdot d\vec{x} + \dots) \\ &= \int \vec{F}_G \cdot d\vec{x} + \int \vec{F}_E \cdot d\vec{x} + \int \vec{F}_{Ff} \cdot d\vec{x} + \dots = W_G + W_E + W_{Ff} + \dots \equiv \Delta KE \end{aligned}$$

$$\boxed{Work_{By F_{net}} = Work_{By Gravitation} + Work_{By Elastic} + Work_{By Friction} + \dots = \Delta KE}$$

Given that we have previously discovered that work done by the net force yields a Point Function, and that this fact is striking if for no other reason than that we will never need to perform that integral again, we might ask the question: "If we perform the work integral on other



force categories such as gravitation , might the result turn out similar”?  
 “Will those integrals yield Point Functions as well”?

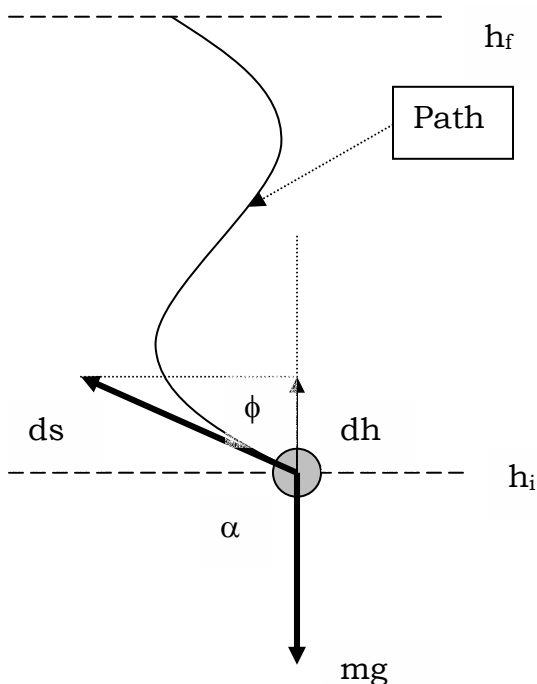
The answer to this question is “Yes, in some cases it will”! It is *that* fact which leads to the remaining energy concepts.

The Work – Energy Theorem master classification scheme is devised as follows:

1. Select a category of force.
2. Apply the work integral to that force.
3. Examine the result...is it a Point Function?
4. If no, go no further.
5. If yes, move it algebraically to the right side of the work energy equation.
6. AND, re-name it as a change in energy.

Path functions on the left of the equals sign, point functions on the right. This is how the remaining configurational energy forms are contrived. We will now embark on that journey.

**Gravitational Potential Energy.** We start with the task, *compute the work done by the force of gravity, near the earths’ surface, to increase an objects altitude, at constant velocity.* We will again consider a single point mass (or center of mass). Of course, the force of gravity alone has never increased an objects altitude, never the less, we may perform the computation. We start by showing an arbitrary path for which the mass will move along. Near the earth, the force of gravity may be expressed as Weight =  $mg$ . We also show a small displacement vector,  $ds$ ; the angle between the force and displacement vector is  $\alpha$ .



Starting with the definition of work,  $Work_{mg} = \int dw$ ,

we first fabricate  $dw$  as

$$dw = |m\vec{g}| \cos \alpha |d\vec{s}|$$

Since  $\alpha$  and  $\phi$  are supplementary angles, we replace the  $\cos \alpha$  with  $-\cos \phi$ ,

$$dw = -|m\vec{g}| \cos \phi |d\vec{s}|$$

Next, we recognize that  $|ds| \cos \phi = dh$  and substitute,

$$dw = -|m\vec{g}| |dh|$$

Notice that the effect of introducing  $dh$  is that no matter how the direction of  $ds$  varies along the path,  $dh$  becomes the vertical projection of  $ds$ . We now apply the integral,

$$Work = \int_{h_i}^{h_f} -|m\vec{g}| |dh|$$

$$Work_{mg} = -|mg|(h_f - h_i) = -|mg|\Delta h$$

We notice that if we impose a “local coordinate system” for the variable  $h$ , upward is positive and downward is negative, that the result will also hold true when the object decreases altitude. We also notice that the result is a Point Function! The work by “ $mg$ ” does not depend on the path by which we change altitude, only the change in altitude. Our final step is to algebraically move the result to the right side of the work energy theorem and give it a new name, *the change in gravitational potential energy*,  $\Delta GPE$ . This leads to a new version of the **Work Energy Theorem**.

$$Work \text{ done by all forces except } mg = \Delta KE + \Delta GPE$$

where,

$$\Delta KE = \frac{1}{2} M(v_f^2 - v_i^2)$$

$$\Delta GPE = |mg|\Delta h$$

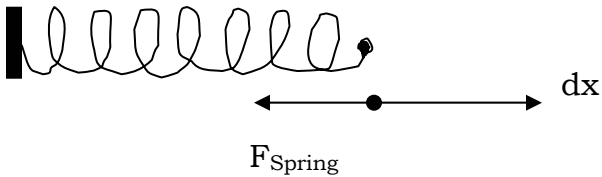
Notice the exclusion of the force “ $mg$ ” from the work done by the net force. This is because “ $mg$ ” work now exists on the other side of the equation under a new name,  $\Delta GPE$ . This quantity also has a new sense; with the sign change, it becomes the work done against (not by) the force of gravity. The work energy theorem has evolved to a new level.

**Elastic Potential Energy.** We start with the task, *compute the work done by the Hooke's law spring force, to move that springs free end away from its relaxed position at constant velocity.* Of course, a spring will not, by itself, move its free end away from the relaxed position. This will not stop us from performing the computation.

We should first remind the reader of the equation relating the force of a Hooke's law spring to its displacement. This equation is:

$$\vec{F}_{Spring} = -k\vec{x}$$

The variable “k” is an empirical positive constant known as the “Hooke's law spring constant”. It is generally obtained through experiment by plotting the spring force as a function of end displacement. This plot produces a straight line of slope “k” and of intercept, zero. Thus, the local variable “x” must be assigned the value of zero when the spring is in its relaxed state.



We apply the work integral,

$$Work_{F_{spring}} = \int \vec{F}_s \cdot d\vec{x}$$

We expand the dot product, noting that the force of the spring is always 180 degrees to that of the displacement.

$$Work_{F_{spring}} = \int_{x_i}^{x_f} |-kx| \cos 180 dx = -|k| \int_{x_i}^{x_f} x dx$$

We complete the integral,

$$Work_{F_{spring}} = -\frac{1}{2}k(x_f^2 - x_i^2)$$

We notice that the outcome of the integral is a Point Function. It depends on the initial and final position of the spring and none of those positions between initial and final. In addition, notice that the plus/minus sign convention for x is not important since the x value is squared. Only the relaxed position need be carefully assigned, to zero. We now apply the final step. Algebraically move the result to the right side of the equation and assign it a new name, *the change in elastic potential energy, ΔEPE.*

**Work Energy Theorem.** Below is our final version of the Work Energy Theorem:

$$Work = \Delta KE + \Delta GPE + \Delta EPE$$

where,

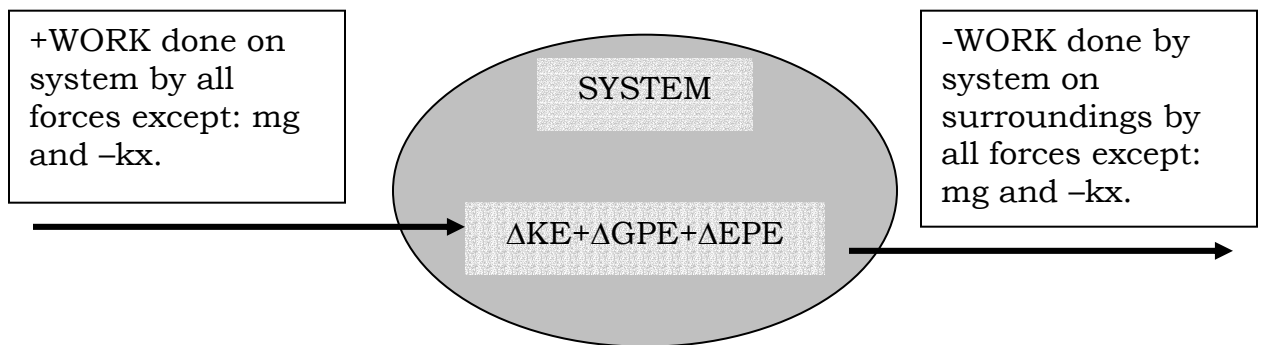
WORK, is the work done by all forces in the net force EXCEPT “mg” and “kx”.

$$\Delta KE = \frac{1}{2}m(v_f^2 - v_i^2)$$

$$\Delta GPE = |mg|\Delta h$$

$$\Delta EPE = \frac{1}{2}k(x_f^2 - x_i^2)$$

and, mass must remain constant, up is positive for “h”, and x = zero when relaxed.



The above schematic diagram represents the accounting basis of the work energy theorem. That, which is put in, minus that which is pulled out, equals the change. With this variation of the work energy theorem, we possess *deposit+*, *withdraw-*, and *account transfer* possibilities.

When applying the work energy theorem to a problem, we also follow a procedure similar to an account audit. The procedure is as follows:

1. **Select an audit time interval, *initial and final*.** Even though the times are not used as variables, it is important to understand that this is an audit over a time.
2. **Evaluate the energy side of the work energy theorem for all objects in the system at the *initial and final* times.** Do not view the energy status anywhere along the path!

Remember, Energy change is a POINT FUNCTION. This is usually the easier part of the audit.

3. **On the work side of the work energy theorem, we ask the question:** “did any external force, not excluded ( $mg$ ,  $kx$ ) from the net force, do work anywhere along the path between initial and final?”. If the answer is yes, we formulate the work integral for that force. This part of the audit is the more difficult. A force vector diagram and a small displacement vector ( $ds$ ) near by will help to think this through. Remember, the concept is to look for a small amount of work,  $dw$ , and then add them up along the path. If the forces involved are constant forces, this procedure is much easier to accomplish.
4. **Finally, we solve the fabricated work energy equation for the desired variable or relationship.**

*What the Work Energy Theorem is NOT.* The student must understand the limitations to the work energy theorem. The concept is SCALAR, not vector. The work energy theorem will never explain why a physical system responds as it does. True explanations lie in the concepts of Force and Motion. Vector quantities. The work energy theorem is only an accounting concept. An accountant can never explain why a business makes or loses money, only where the money went and how much; only the business owner can explain why. Similarly, only the physicist, using Force and motion can explain why a system responds as it does. The work energy theorem is however an extremely powerful tool to do physics. It will lead to relationships useful in the solution of many problems.

**The law of conservation of energy.** It is usually stated that: *for the system of the universe, energy can neither be created nor destroyed (except in a nuclear reaction).* Energy is conserved! Whether we choose to debate this statement or not, let us understand one important point:

**The system of the Universe is all but useless to the solution of real problems.** For the system of the universe, all forces are internal; they add as vectors to exactly ZERO. The solution of real problems requires system boundaries significantly smaller than the universe. Therefore, one must for these smaller systems, carefully examine them for external forces capable of doing work. For systems smaller than the universe, useful systems, Energy is not necessarily conserved.

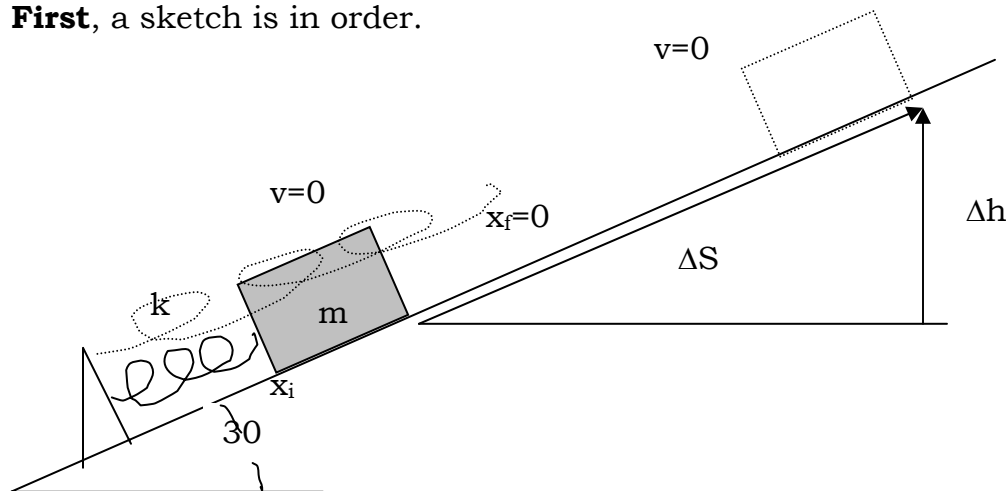
### Example Application.

An object of mass 2 kg is placed on an incline of 30 degrees where on, a 400 n/m spring is attached. The object is pushed down on the spring such that the spring compresses .2 meters (spring not attached to the mass) and then released. There is friction between the mass and incline ( $\mu_k = .3$ ). How far up the incline, measured from the release position, will the mass travel?

### Solution.

The solution to this problem could be accomplished entirely using force and motion equations; the student should consider attempting it in that manner. Here, we will attempt the solution using the Work Energy Theorem.

**First**, a sketch is in order.



We show the block and spring in the compressed state at time *initial*. Also shown is the block and spring (dotted) at time *final*. The value  $\Delta S$  has been assigned to the total displacement that the block will traverse.

**Second**, we introduce the form of the work energy theorem which will be used for the solution.

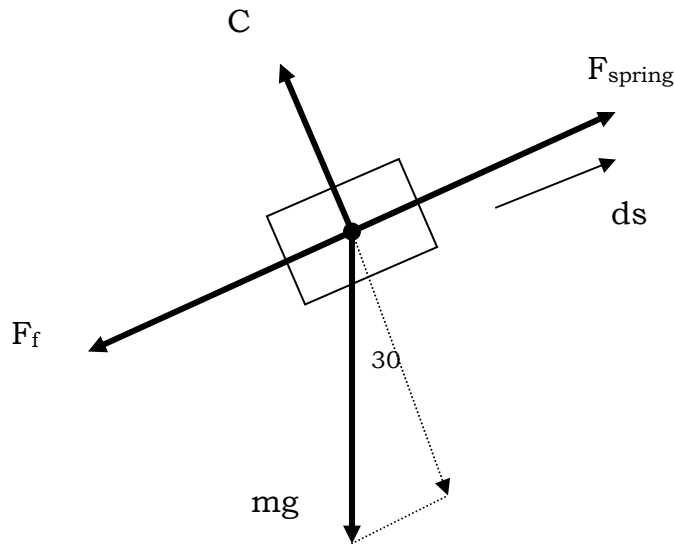
$$\text{Work}_{\text{By all except Mg, } kx} = \Delta KE + \Delta GPE + \Delta EPE$$

With the form selected, we evaluate the energy changes from initial to final. For the kinetic energy,  $v_{\text{initial}}$  and  $v_{\text{final}}$  are identical, both are zero, so there is no change in kinetic energy; this term will be zero. This is the beauty of a point function! Even if the values were not zero, if they are the same at initial and final, the kinetic energy change would be zero.

For the gravitational potential energy, there is an increase in altitude, as shown by  $\Delta h$  in the diagram; this term will be non zero.

For the elastic potential energy, the free end of the spring is initially compressed, and at time final it is at its relaxed state; this term will also be non zero.

**Third**, we ask the question “did any external and non excluded force do work anywhere along the displacement path”? To answer the question we will construct a force vector diagram.



The surface compression force, “C”, is 90 degrees to the displacement vector and thus does no work. Both the weight vector, “mg”, and the force of the spring, “F<sub>spring</sub>” are excluded from the work side of the equation. This leaves the kinetic friction force and it *does* work.

We will need to express this friction force in terms of the compression and the coefficient of friction.

$$F_f = \mu_k C$$

and from noting that the acceleration perpendicular to the incline is zero, we have the compression force “C” in balance with the perpendicular component of the weight vector, “mg Cos 30”.

$$F_f = \mu_k mg \cos 30$$

We also notice that this force is constant so the integral form of work is not necessary.

**Finally**, we assemble the work energy theorem:

$$\text{Work}_{\text{By all except Mg, kx}} = \Delta \text{KE} + \Delta \text{GPE} + \Delta \text{EPE}$$

$$|F_f| |\Delta S| \cos 180 = |mg| \Delta h + \frac{1}{2} k (x_f^2 - x_i^2)$$

$$|\mu_k mg \cos 30| |\Delta S| \cos 180 = |mg| \Delta h + \frac{1}{2} k (x_f^2 - x_i^2)$$

The work energy theorem is thus applied. To solve for  $\Delta S$ , we require a substitution for  $\Delta h$ . From trigonometry,  $\Delta h = \Delta S \sin 30$ . We substitute this relationship along with the value of  $x_f$  (which is zero).

$$|\mu_k mg \cos 30| \Delta S |\cos 180| = |mg| \Delta S \sin 30 + \frac{1}{2} k (0 - x_i^2)$$

We are now in a position to solve the expression for  $\Delta S$ .

$$\Delta S = \frac{kx_i^2}{2|mg|(\sin 30 + \mu_k \cos 30)}$$

$$\Delta S = \frac{400 \text{ n/m } (.2 \text{ m})^2}{2 |2 \text{ kg } 9.8 \text{ m/s}^2| (\sin 30 + .3 \cos 30)} = .537 \text{ meters}$$