

Rotational Dynamics of Rigid Solids

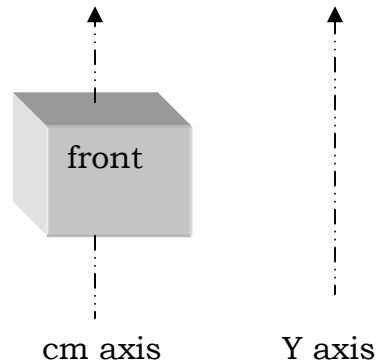
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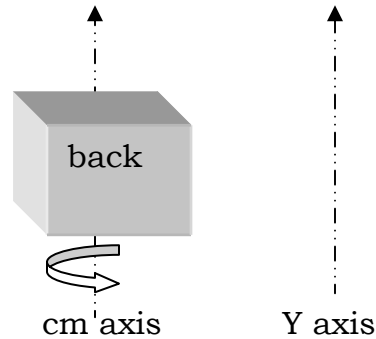
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Rotational Dynamics of Rigid Solids

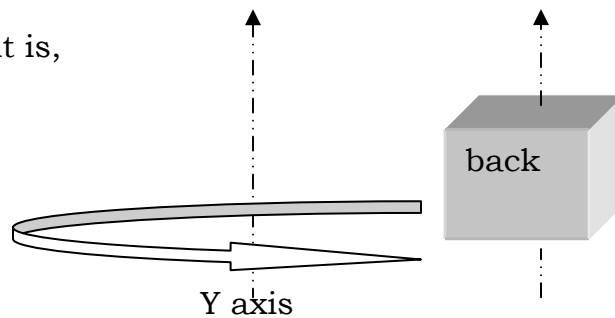
1 Introduction. It is this author's intent to start with the laws of Newton, applied to a collection of particles, and deduce all laws of rotation. To provide a crystal clear path from one to the next. In consideration of this journey, the student must be aware that the basis for true understanding and clarity is as much a matter of perspective and interpretation, as it is an exercise in mathematics and the application of fundamental laws of nature.



Consider the above situation: We have a rigid block. We give it a 180 degree rotation about each of the two axes shown, the center of mass axis, and the Y axis which is parallel to the cm axis. For the cm axis, the result is,



and for the Y axis, the result is,

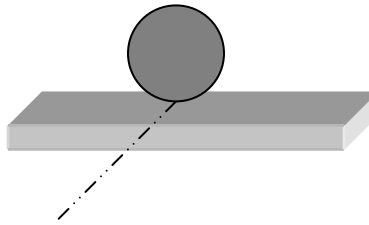


We compare the result of each operation. Many observers would consider the results as dissimilar – truly visual inspection would agree. The cm

rotation produces no translation of the center of mass of the box whereas the Y rotation produces considerable displacement of the center of mass of the box. And yet, BOTH were 180 degree rotations. A matter of perspective and interpretation.

We require a distinction between position, and orientation. While the change in *position* of the box is different for each event, both events show the same change in *orientation* of the box. The orientation has changed by 180 degrees. We will designate the objects cm *position* with the usual position vector \mathbf{R}_{cm} , and define the objects *orientation* as its *angular position* θ . It can also be seen from the example that a change in angular position (a rotation) about a given axis is equivalent to a change in angular position (a rotation) about any other *parallel axis*.

It is important to create a clear understanding of the above paragraph here and now. We often use words such as “rotate” and “spin”; and in most cases our mind associates an axis through the body of the object. When we say “rotate” or “spin”, we must THINK change in orientation – change in angular position. We must also be cautioned to guard against our mind’s selection of axis. Consider the following question: *Is a ball rolling along on a flat table spinning about an axis which passes thru the point of contact?*



The answer is of course, Yes!

We now define the quantities of angular motion. The angular position variable (orientation) is defined as θ . The angular velocity variable (the rate of change of angular position) is defined as ω . The angular acceleration variable (the rate of change of angular velocity) is defined as α . The defining relationships are:

$$\omega \equiv \frac{d\theta}{dt}$$

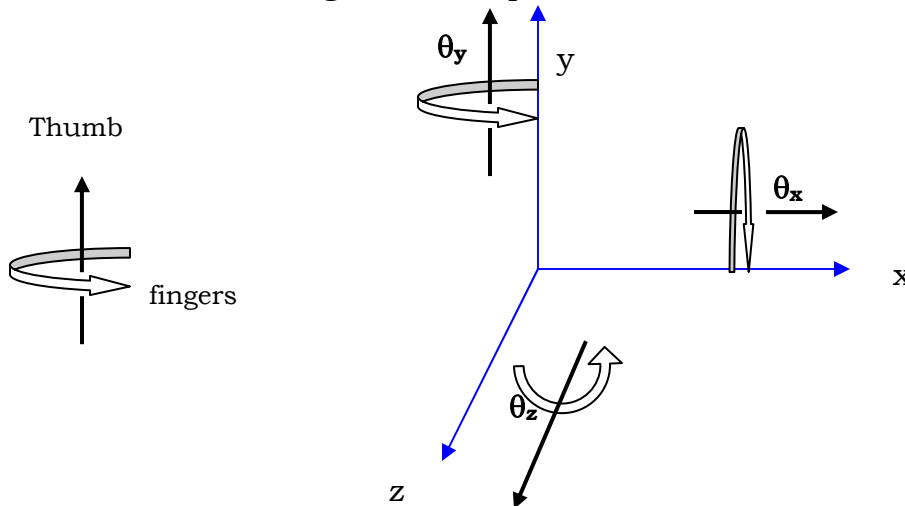
$$\alpha \equiv \frac{d\omega}{dt}$$

Notice the similarity to the translational variables, velocity and acceleration.

Consider a new question. We observe a point mass particle moving through space. We observe it for only an infinitesimally short time. *Is the particles' velocity translational or angular?* It would seem that the question is impossible to answer. If we were able to observe the particle at other times, before and after the given observation, we may feel we are in a better position to answer the question. The answer is however a matter of perspective and interpretation! We have the ability to describe the motion of the particle as *either* translational, or angular. It is a matter of choice. We often make this choice as a result of other information, its' history and future, however the choice can not be wrong, only inconvenient. It is exactly for this reason, convenience, that the angular quantities are conceived.

We have provided the mind set for the discussions to follow, we next present the tools.

2. Coordinate system. Of the many coordinate systems available, mathematics and engineering tend to favor the “right hand” coordinate system. Positive angular quantities are found by placing the thumb in the direction of positive on the translational axis (right hand) and observing the direction of the fingers. To represent the angular quantity as a vector, we think “thumb”, not “fingers”. This preserves the vector icon of the arrow.

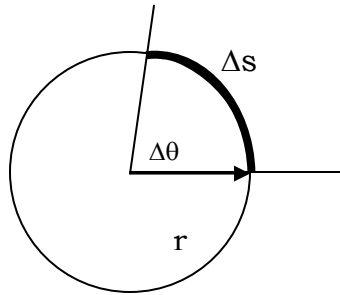


It must also be remembered that the choice of coordinate system is ultimately that of the problem solver. The above coordinate system is not mandatory, it will however provide a basis of communication.

3. Circular to Angular Transformations. Three circular to angular transformations will be used to make the connection between the translational view and the angular view. Their basis lies in the fact that we

can easily view the circular motion of a particle as both translational and angular.

Consider a particle which is displaced along a circular path of radius r , by an amount Δs .



This displacement could also be interpreted as an angular displacement of $\Delta\theta$ about an axis thru the center of that circle directed out from the page. Using the formula for the length of a circular sector from geometry, we have

$$\Delta S = \Delta\theta r$$

where the angular displacement MUST be measured in radians.

We now consider the velocity of our particle. By differentiating the above relationship with time, for constant radius, we have

$$\frac{dS}{dt} = \frac{d\theta}{dt} r$$

We recognize the term on the left of the equal sign, ds/dt , as the speed of the particle along the path, the tangential velocity, v_t . The derivative on the right side of the equal sign, $d\theta/dt$, is also recognized as the angular velocity of the particle about any "out of the page" axis, ω . This yields the second transform,

$$v_t = \omega r$$

The third and final transform is obtained from the above in like manner,

$$\frac{dv_t}{dt} = \frac{d\omega}{dt} r$$

$$a_t = \alpha r$$

where a_t is the acceleration component tangent to the circle of motion, and α is the angular acceleration of the particle about any axis perpendicular to the plane of the circle.

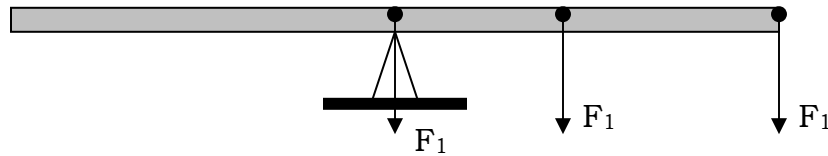
$$S = \theta r$$

$$v_t = \omega r$$

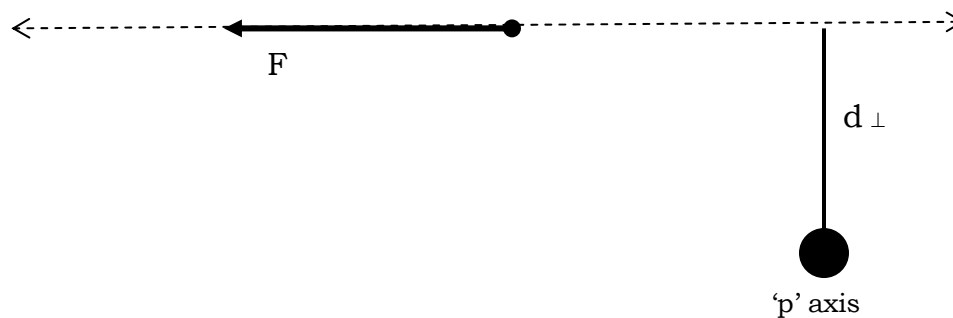
$$a_t = \alpha r$$

All angular quantities require radian angle measure. V_t and a_t are measured in the "r=0" frame of reference. The Vector aspects of these equations will be discussed later.

4. The Cause Variable. We start with the question, “why do objects angularly accelerate”? To answer this question, let us consider a simple “see saw” (a rigid board balanced on a fulcrum).



We consider the effect of the application of the force F_1 at various places along the board. When applied to the center (at the fulcrum) we observe nothing. The fulcrum will push up on the board with a compression force to maintain equilibrium. It is when we move the force application point toward the end of the board that we notice a rotation about the fulcrum; a change in the angular velocity of the board. An angular acceleration. Experience suggests that as either the force value is increased, or the application point is moved further from the fulcrum, the angular acceleration increases. The cause of angular acceleration involves the force magnitude, the forces direction, and the location of application. We will call this Cause variable *TORQUE*.



The Torque, due to force F , about axis ‘p’ (out of the page), is found in the following procedure:

1. Extend the “line of action” of the force.
2. Locate and measure the perpendicular distance from the line of action of the force, back to the axis. “ d_{\perp} ” must be both perpendicular to the “line of action” and the axis.
3. Compute the magnitude of the Torque as: $\tau = d_{\perp} F$
4. Assess the direction of the angular acceleration; this is also the direction of the Torque. This can be done by physically holding down the paper at axis ‘p’ and pulling on the paper at the force application point. In the diagram above, the direction would be counter clock wise (or out of the page if you think “thumb” instead of “fingers”).

Granted, the above procedure seems crude; it is. It does however introduce at a fundamental level, what Torque is. There are some important points to emphasize here.

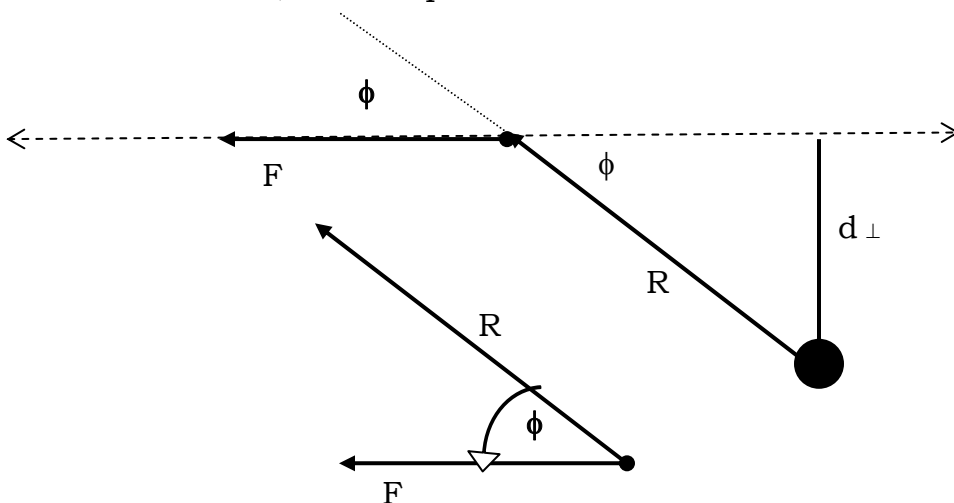
- The exact coordinate of the application of the force must be determined.
- The translational coordinate system must be specified. Choosing the axis is in effect choosing both the axis direction and the origin of the coordinate system.
- The magnitude and direction of the Torque will depend on the coordinate system axis. Note that if the line of action should go through the axis, d_{\perp} will be zero, and thus the Torque about that axis will also be zero.
- The direction of the torque (“thumb”) is always perpendicular to the plane defined by the force vector and d_{\perp} (in the direction defined by the axis).
- When considering Torque, the sentence: “*Torque due to _____, about _____ axis.*” should be in ones’ mind. The first blank indicating the Force, or Force collection, and the second blank indicating the axis for the computation.

This procedure has been formalized as a mathematical operator, similar to the DOT Product (scalar product) used to compute work. The operator is called a CROSS Product (vector product). The notation is:

$$\vec{\tau} = \vec{R} \times \vec{F}$$

where R is a location vector identifying the location of the force application point in the chosen coordinate system.

Using the location vector, it is easy to show that $d_{\perp} = R \sin \phi$, where ϕ is the angle acquired when the location vector and the force vector are positioned tail to tail, R is swept into F .



Torque is the CAUSE variable; the cause of angular acceleration.

5. Mathematical Interlude. At this point, we present the tools of Vector Arithmetic. A right hand coordinate system is assumed. The presentation is done without derivation. Numerous texts on the subject may be consulted by the student unfamiliar with the topic. Here, we will only remind the student of their existence.

The Dot Product or scalar product as it is often called was used in the definition of work. It is:

$$\vec{A} \bullet \vec{B} \equiv (A_x B_x) + (A_y B_y) + (A_z B_z) = |\vec{A}| |\vec{B}| \cos \phi_{A \rightarrow B}$$

The Cross Product or vector product as it is often called was used in the definition of Torque. It is possible execute the cross product operator as a pure vector operation using i, j, k notation. This is accomplished by the evaluation of a 3 by 3 determinate, we also expand by minors:

$$\vec{A} \times \vec{B} \equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y) \vec{i} - (A_x B_z - A_z B_x) \vec{j} + (A_x B_y - A_y B_x) \vec{k}$$

The magnitude of the cross product may be computed as:

$$\|\vec{A} \times \vec{B}\| = |\vec{A}| |\vec{B}| \sin \phi_{A \rightarrow B}$$

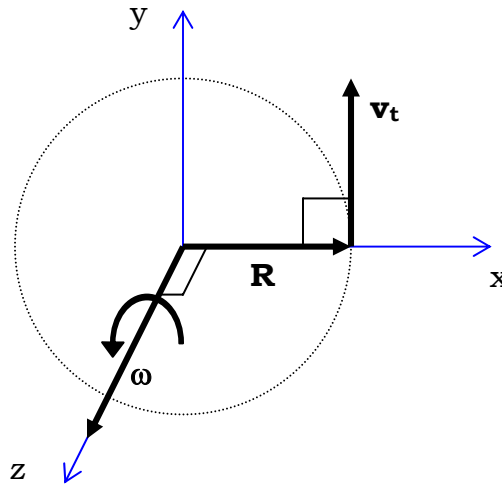
The Cross Product is *not commutative!*

Remember however, given all the fancy notation above, the fundamental meaning of $\vec{A} \times \vec{B}$ is: extend the line of action of the B vector (at location A), determine the perpendicular distance from that line of action to the axis, multiply B by that distance, and assess the direction as along the axis (+ or -).

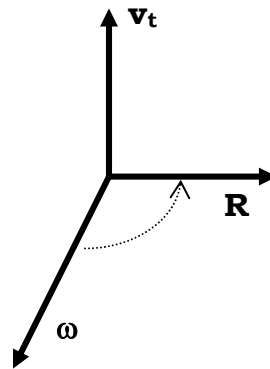
Relationships using these two operators are listed below.

1. $\vec{A} \bullet \vec{A} = A^2$
2. $\vec{A} \times \vec{A} = 0$
3. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$, not commutative, the order is important.
4. $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$, distributive.
5. $\vec{A} \bullet (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \bullet \vec{C} = \vec{B} \bullet (\vec{C} \times \vec{A}) = \text{scalar}$
6. $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \bullet \vec{C}) - \vec{C}(\vec{A} \bullet \vec{B}) = \text{vector}$

To complete our mathematical interlude, we return to the circular to angular transformations. These relationships, previously presented as scalar relationships, can now be interpreted as vector relationships.



Consider a particle moving in a circular path of radius R at a speed v_t . Examination of the three vectors, shows that their directions are all perpendicular to each other.



Crossing ω into \mathbf{R} will result in a vector perpendicular to the plane defined by ω and \mathbf{R} , \mathbf{v}_t . Thus, we may describe each of the circular to angular transformations using the Cross Product operator.

$$\begin{aligned}\vec{S} &= \vec{\theta} \times \vec{R} \\ \vec{v}_t &= \vec{\omega} \times \vec{R} \\ \vec{a}_t &= \vec{\alpha} \times \vec{R}\end{aligned}$$

A right hand coordinate system is assumed. v_t and a_t are in the frame of reference of " $R=0$ ". This concludes our mathematical interlude.

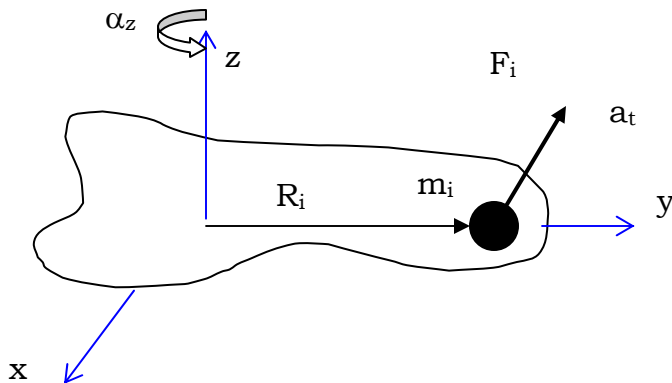
6. Angular Relationships; The Strategy. Prior to developing the angular relationships, it is prudent to review the reasons we do so. The reason is convenience. We may always view an object as a large collection of small “point masses”; this view is what we will call the MICRO view. The MICRO view is sufficient to understand all properties of motion of an object. Using the MICRO view to compute motion properties is however very cumbersome. It requires many equations, and tedious mathematics. If there should exist a single relationship, capable of similar computational power, it would seem wise to search it out. If found, this equation will be considered a MACRO view. This then outlines the strategy we will follow.

For the case of Vector relationships, there is a “trick” which we will commonly use, we will Cross Product the translational equation by a location vector from the left, followed by a series of simplifications and interpretations.

For the case of Scalar relationships, we will perform a simple summation, followed by simplification and interpretation.

In all cases, we will start with a physical relationship well founded in science.

7. Cause and Effect. Consider a collection of particles forming a rigid solid, constrained to rotate about the z axis.



We start by applying $F=ma$ to this particle at the instant shown, in the direction tangent to its motion in the x – y plane.

$$\vec{F}_i = m_i \vec{a}_{i,t} \quad (\text{single point})$$

Next, we cross product both sides of the above equation by the particles location vector.

$$\vec{R}_i \times \vec{F}_i = \vec{R}_i \times (m_i \vec{a}_{i,t})$$

The left side of the above equation is now interpreted as the Torque, due to F_i , about the z axis.

$$(\vec{\tau}_z)_i = \vec{R}_i \times (m_i \vec{a}_{i,t})$$

Next, the scalar m_i is factored out of the right side of the equation and the third circular to angular transformation replaces a_t .

$$(\vec{\tau}_z)_i = m_i \vec{R}_i \times (\vec{\alpha}_z \times \vec{R}_i)$$

The triple Cross Product is next expanded using eq. 6,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}),$$

$$(\vec{\tau}_z)_i = m_i [\vec{\alpha}_z (\vec{R}_i \cdot \vec{R}_i) - \vec{R}_i (\vec{R}_i \cdot \vec{\alpha}_z)]$$

We are now in a position to interpret, $\vec{R}_i \cdot \vec{R}_i = R_i^2$ which is scalar, and

$\vec{R}_i \cdot \vec{\alpha}_z = 0$ since R and α are at 90° to each other.

This yields the result,

$$(\vec{\tau}_z)_i = m_i R_i^2 \vec{\alpha}_z$$

If we were to perform the above derivation on each and every particle, and add the individual results, we would arrive at

$$\sum_{i=1}^n (\vec{\tau}_z)_i = \sum_{i=1}^n m_i R_i^2 (\vec{\alpha}_z)_i$$

Finally, for a rigid collection of points, all α_i are identical and may be factored out of the summation giving

$$\sum \vec{\tau}_z = \left(\sum_{i=1}^n m_i R_i^2 \right)_z \vec{\alpha}_z \quad (\text{fixed axis})$$

We are now in a position to interpret and summarize the above equation.

- $\sum \vec{\tau}_z$ represents the net Torque, due to all external forces, about the z axis. It is the Cause Variable.
- $\vec{\alpha}_z$ is the resulting angular acceleration of the rigid solid about the z axis. It is the Effect Variable.
- $\sum m_i R_i^2$, a scalar, must be the objects resistance to being angularly accelerated about the z axis. Each R_i is the location vector, measured perpendicular to the z axis, of each particle in the object. We will provide a new name and variable for this macro quantity, **moment of inertia, I_z** about the z axis.

Our Macro equation becomes,
$$\sum \vec{\tau}_z = I_z \vec{\alpha}_z \quad (\text{rigid body, fixed axis})$$

8. Moment of Inertia, a Macro Quantity. We have coined the term “moment of inertia” and assigned the meaning, *resistance to angular acceleration*. It is interesting to note that the word “moment”, which we now associate with a time interval, was in fact defined as a “rotation”. When we say “wait a moment”, we are in fact saying “wait for a rotation of the shadow on a sun dial clock”!

The moment of inertia of a point mass about an axis is:

$$I_{point} = m R_{\perp}^2$$

where R_{\perp} is the distance, measured perpendicular to the axis, from the axis to the point.

For a collection of points in a rigid solid, their moment of inertia is given by:

$$I_{collection} = \sum m_i R_{\perp i}^2$$

The above summation can be performed over a rigid body as an integral to compute the moment of inertia of that body about a given axis:

$$I_z = \int R_{\perp z}^2 dm$$

The above computation maybe performed for a wide variety of objects, about various axis', and tabled as MACRO quantities for the convenience of later use.

As a parting thought, we remind the student that the moment of inertia is not shape dependent, it depends on how the objects mass is distributed as viewed from the chosen axis.

9. Newton's Second Law for Rotation. In the previous section, Cause and Effect, we developed the angular view of $F=ma$. We understand that Newton's second law was stated in a much more general way, utilizing the concept of momentum. We will here perform much the same derivation as in the Cause and Effect section, on Newton's second law. Consult the figure there for clarity.

Newton's second law for a particle is:

$$\vec{F}_i = \frac{d(m\vec{v})_i}{dt}$$

We will assume that F is an external force acting in the $x - y$ plane on a particle at position R . We Cross Product the above equation, on the left, by the position vector R .

$$\vec{R}_i \times \vec{F}_i = \vec{R}_i \times \frac{d(m\vec{v})_i}{dt}$$

We recognize the left side of the equation as the Torque, due to F, about the z axis on the particle. On the right side of the equation, we move the constant R, into the derivative.

$$(\vec{\tau}_z)_i = \frac{d\{\vec{R}_i \times (m\vec{v})_i\}}{dt}$$

It is useful at this time, to assign a new variable to the quantity within the brackets,

$$\vec{l} \equiv \{\vec{R} \times (m\vec{v})\},$$

whose name will be *angular momentum*. The result is Newton's second law for angular motion:

$$\vec{\tau} \equiv \frac{d\vec{l}}{dt}$$

While the definition of angular momentum, $\vec{l} \equiv \{\vec{R} \times (m\vec{v})\}$, may be useful for a single particle (or the center of mass particle), we will continue to expand it by substituting the second circular to angular transform.

$$(\vec{\tau}_z)_i = \frac{d\{\vec{R}_i \times m[\vec{\omega}_z \times \vec{R}_i]\}}{dt}$$

We factor out the scalar, m, and expand the triple cross product,

$$(\vec{\tau}_z)_i = \frac{d\{m(\vec{R}_i \times [\vec{\omega}_z \times \vec{R}_i])\}}{dt}$$

$$(\vec{\tau}_z)_i = \frac{d\{m(\vec{\omega}[\vec{R}_i \cdot \vec{R}_i] - \vec{R}_i[\vec{R}_i \cdot \vec{\omega}_z])\}}{dt}$$

We are now in a position to interpret, $\vec{R}_i \cdot \vec{R}_i = R_i^2$ which is scalar, and $\vec{R}_i \cdot \vec{\omega}_z = 0$ since R and ω are at 90° to each other. The resulting outcome is:

$$(\vec{\tau}_z)_i = \frac{d\{mR_i^2\vec{\omega}_z\}}{dt}$$

$$\sum (\vec{\tau}_z)_i = \frac{d\{\vec{\omega}_z \sum [mR_i^2]\}}{dt} = \frac{d\{I_{\perp z}\vec{\omega}_z\}}{dt}$$

and it can be seen that the angular momentum of a rigid solid can also be expressed as:

$$\vec{l} \equiv I_{\perp} \vec{\omega} \quad (\text{Rigid solid, fixed axis})$$

10. Angular Impulse and Momentum. As one may suspect, one may reinvent Newton's law for angular motion, and move it into a very useful form. Starting with

$$\vec{\tau} \equiv \frac{d\vec{l}}{dt}$$

we multiply both sides by dt, and integrate.

$$\vec{\tau} dt = d\vec{l}$$

$$\int \vec{\tau} dt = \int d\vec{l}$$

$$\int \vec{\tau} dt = \Delta\vec{l} = \Delta(I\vec{\omega}) \quad (\text{fixed axis})$$

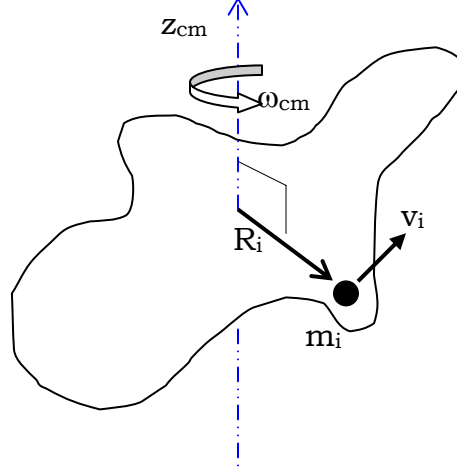
Application of angular impulse and momentum is in many ways similar the translational version. One interesting exception, is in the treatment of the Torque. With clever selection of axis, it is often possible to make the torque due to an external force equal zero and further simplify the application.

11. Energy Relationships; The Strategy. The total kinetic energy of any collection of particles, is the sum of the individual kinetic energies of the individual particles. Here in lies our basic strategy. There are however a few cautions which must be considered once a Macro form of that total energy is developed. Kinetic energy is frame of reference dependent in that the underlying velocity (speed) is itself dependent on the frame of reference from which it is measured. Work also depends on the frame of reference in that the displacement is depends on the frame of reference.

For example, a ball rolling across a table fixed to the earth will have kinetic energy in the earth frame of reference, whereas in the ball's center of mass frame of reference, the balls velocity and kinetic energy will be zero. This is not an energy violation; it is a mater of view. The energy distribution in a system, and what forces may or may not do work is dependent on the frame of reference.

When doing any energy audit, the student must clearly state the frame of reference, and then consider the forms and amounts of energy involved; taking care not to ignore an amount of energy, nor collect a single amount of energy twice!

12. Rotational Energy; A Macro Form. We start with a collection of identical particles which form a rigid solid, rotating about the z_{cm} axis (which is also the collections center of mass) as shown.



We compute the i^{th} particles kinetic energy from the frame of reference of the particles center of mass.

$$KE_i = \frac{1}{2} m_i v_i^2$$

For all particles, the total kinetic energy is the sum the individual kinetic energies.

$$KE_{total} = \sum KE_i = \sum_{i=1}^n \frac{1}{2} m_i v_i^2$$

We now examine the second circular to angular transformation to arrive at a substitution for v^2 .

$$v_i^2 = \vec{v}_i \cdot \vec{v}_i = (\vec{\omega}_i \times \vec{R}_i) \cdot (\vec{\omega}_i \times \vec{R}_i) = \omega_i^2 R_i^2$$

We substitute in the kinetic energy equation,

$$KE_{total} = \sum_{i=1}^n \frac{1}{2} m_i \omega_i^2 R_i^2$$

Note that for a rigid solid, although the v_i are not identical, $\omega_1 = \omega_2 = \omega_3 \dots = \omega_{cm}$. We factor out of the summation, $\frac{1}{2}$, and ω_{cm} .

$$KE_{total} = \frac{1}{2} \left(\sum_{i=1}^n m_i \vec{R}_i^2 \right) \omega_{cm}^2$$

The quantity within the parentheses is recognized as the objects moment of inertia about the cm axis. We now define the Macro Rotational Energy as:

$$RE \equiv KE_{total} = \frac{1}{2} I \omega^2$$

The above rotational energy is in fact the micro kinetic energy of the collection of particles in the rigid solid, rotating about the axis for which the moment of inertia (and angular velocity) is computed.

13. Work; An Alternative View. Contemplate the task of starting a power lawn mower. In pulling the mower's cord, we apply a force over a displacement and thus do work. We may however wish to view this task in another way; we apply a torque by means of the tension in the cord and angularly displace the shaft of the motor. Both views are of the same work. Only the view of it is different. We derive the equivalent.

$$Work \equiv \int \vec{F} \cdot d\vec{x}$$

Choosing our axis origin through the shaft of the motor, we both multiply and divide the force by the perpendicular distance from the shaft axis to the line of action of the force, d_{\perp} .

$$Work \equiv \int \frac{d_{\perp}}{d_{\perp}} \vec{F} \cdot d\vec{x}$$

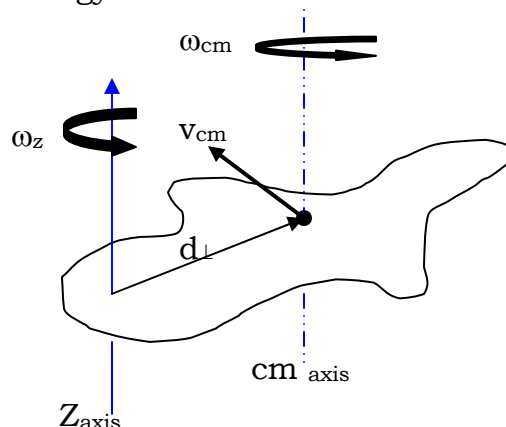
The product $d_{\perp} F$ is the torque due to the force, about the axis. The quantity dx / d_{\perp} is the angular displacement, $d\theta$. Substitution gives,

$$Work \equiv \int \vec{F} \cdot d\vec{x} \equiv \int \vec{\tau} \cdot d\vec{\theta} \quad (\text{fixed axis})$$

We again remind the reader that these are not two separate works, only two alternative views of the same work.

14. Parallel Axis Theorem. Consider a collection of particles which form a rigid solid, rotating about a fixed z axis (not through the center of mass) as shown below. We may audit the objects energy of motion from the frame of reference of this fixed z axis in a number of ways. First, as in the previous section, we can accept the Micro view; the sum of all the individual kinetic energies of the individual particles. Second, we can accept the Macro view; the rotational energy, $\frac{1}{2} I \omega^2$, about the z axis.

There is however a third "*combination*" view of the objects energy which is often quite useful. In the combination view, we add the kinetic energy of the objects center of mass as if it is a single particle, to the objects rotational energy about the objects center of mass axis which is parallel to the z axis. See Appendix II. All three views of the total energy are identical; they produce the same total energy.



We equate the pure rotational view of the energy about the z axis to that of the combination view:

$$\begin{aligned} \frac{1}{2} I_z \omega_{z \text{ axis}}^2 &\equiv KE_{cm} + RE_{cm \text{ axis}} \\ \frac{1}{2} I_{z \text{ axis}} \omega_{z \text{ axis}}^2 &= \frac{1}{2} m_{\text{total}} v_{cm}^2 + \frac{1}{2} I_{cm \text{ axis}} \omega_{cm \text{ axis}}^2 \end{aligned}$$

We may use the second circular to angular transform to replace v_{cm} in the above equation. $v_{cm}^2 = \omega_z^2 d_{\perp}^2$.

$$\frac{1}{2} I_{z \text{ axis}} \omega_{z \text{ axis}}^2 = \frac{1}{2} m_{\text{total}} \omega_{z \text{ axis}}^2 d_{\perp}^2 + \frac{1}{2} I_{cm \text{ axis}} \omega_{cm \text{ axis}}^2$$

Recalling that an angular displacement about an axis is equivalent to the same angular displacement about any parallel axis, $\omega_{z \text{ axis}} \equiv \omega_{cm \text{ axis}}$, we cancel both $\frac{1}{2}$ and ω_z from the above equation to give the Parallel Axis Theorem.

$$I_{z \text{ axis}} = m_{\text{total}} d_{\perp}^2 + I_{cm \text{ axis}}$$

The above derivation not only illustrates various perspectives and interpretations of rotational energy, but provides a most useful formula for computing the moment of inertia of a rigid body. Most tables of moment of inertia list those values for axis' through the center of mass of the object. The parallel axis theorem provides the to shift the axis of computation to other locations.

15. Rotational Equilibrium. Beginning students of physics are often introduced to the concept that when an object is at rest,

$$\sum \vec{F}_{\text{external}} = 0 \text{ and } \sum \vec{\tau}_{\text{external}} = 0$$

The above being described as the first, and second, condition of equilibrium.

Nearly all the examples presented the student involve objects at rest, that is zero translational acceleration and zero angular acceleration. We now wish to consider the case of non-zero translational acceleration. “*Is an object, whose center of mass is accelerated, but is not spinning otherwise, angularly accelerated?*” As for example, a car on a level road, which is applying its brakes to stop; is the car angularly accelerated? Before we formulate an answer, let us look at $F=ma$ applied to the center of mass of the object.

$$\sum \vec{F}_{\text{ex}} = m_{\text{total}} \vec{a}_{cm}$$

We cross product both sides of this equation by the instantaneous location vector of the center of mass,

$$\vec{R}_{cm} \times \sum \vec{F}_{\text{ex}} = \vec{R}_{cm} \times m_{\text{total}} \vec{a}_{cm}$$

The quantity on the left of the equal sign is identified as the sum of the external torques,

$$\sum \vec{\tau}_{ex} = \vec{R}_{cm} \times m_{total} \vec{a}_{cm}$$

We arrive at the result,

$$\sum \vec{\tau}_{ex} = m_{total} (\vec{R}_{cm} \times \vec{a}_{cm})$$

With the above relationship, we now answer the question; It is a matter of perspective and interpretation – it is a matter of coordinate system!

If the axis origin is located at the object's center of mass (or anywhere along the line of action of the acceleration vector \vec{a}_{cm} , the cross product, $\vec{R}_{cm} \times \vec{a}_{cm} = 0$ and we may conclude that the angular acceleration about that axis is zero. This also leads to the fact that the sum of the Torques about that axis will also be zero.

If however, the axis origin is located any where else, the cross product is not zero; and with substitution of the third circular to angular transformation $\vec{a}_i = \vec{\alpha} \times \vec{R}_{cm}$, followed by the expansion of the triple cross product (similar to that used in the section – Cause and Effect), we arrive at

$$\sum \vec{\tau}_{ex} = m_{total} (\vec{R}_{cm} \times \vec{a}_{cm}) = m_{total} R_{cm}^2 \vec{\alpha}$$

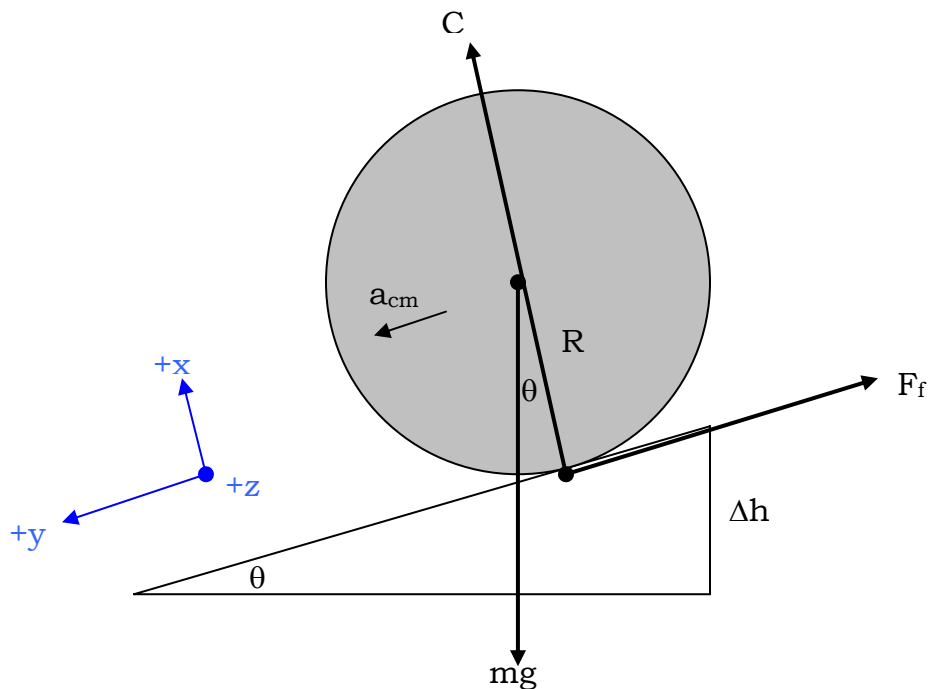
where it is clear that the object possesses a non-zero angular acceleration.

Perspective and interpretation are always considered from a previously selected coordinate system's frame of reference.

16. Examples.

A solid sphere of mass .5 kg and radius .4 m is released ($v_i = \omega_i = 0$) on a 10 degree incline and rolls down *without slippage*. We wish to determine the acceleration of the center of mass of the sphere, the final velocity of the center of mass of the sphere when it's altitude is lowered by 2 meters, and the corresponding angular values as well.

We start with the drawing below. A force analysis results in three forces externally acting on the sphere: the weight of the sphere placed at its center of mass, the compression and friction between the incline and sphere acting at the point of contact. Coordinate system directions for translation are chosen; we will hold off the decision as to where the origin is located momentarily. We can see that the lines of action of all forces do not intersect at a single point in space; this lets us know that the problem will involve angular components.



Finding a_{cm} , solution 1. We start with a choice of origin; it will be placed at the point of contact of the sphere and the incline. Z axis out of the page. The coordinate system is thus fixed to the earth frame of reference and instantaneously at rest. About this axis, the torque due to both C and F_f will be zero. This is clever for two reasons: we do not know the value of C , and we do not know the value of F_f (remember that $F_f = \mu C$ is invalid since

there is no slippage). The only torque producing force is mg , and this we know.

We perform a torque analysis, $\tau = I\alpha$ about this axis. The perpendicular distance from the z axis to the line of action of the force mg is $d_{\perp} = R \sin \theta$, and thus the torque due to mg about the z axis is $\tau = d_{\perp} mg = R \sin \theta mg$.

Looking up the moment of inertia for a sphere we find $I_{cm} = \frac{2}{5} mR^2$. Since our axis is the z axis, not the center of mass axis, we use the parallel axis theorem to offset the axis a displacement of one radius.

$$I_z = I_{cm} + mR^2 = \frac{2}{5} mR^2 + mR^2 = \frac{7}{5} mR^2$$

The equation, $\tau = I\alpha$, is applied.

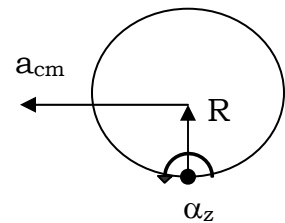
$$\begin{aligned} \tau_z &= I_z \alpha_z \\ R \sin \theta mg &= \left(\frac{7}{5} mR^2\right) \alpha_z \end{aligned}$$

The mass is canceled, and the radius also giving

$$\frac{5}{7} g \sin \theta = R \alpha_z$$

Finally, we recognize the quantity on the right as a_{cm} .

$$a_{cm} = \frac{5}{7} g \sin \theta$$



Note that the outcome does not depend on radius or mass.

Finding a_{cm} , solution 2. In this solution approach, we place our z axis origin at the center of mass of the sphere for the torque computations only. This means that the z axis is accelerated in the earth frame of reference, but is at rest in the center of mass frame of reference. The forces C and mg produce zero torque about the center of mass axis; the force of friction produces a torque of the amount, $R F_f$. The required moment of inertia is about the center of mass axis.

$$\begin{aligned} \tau_{cm} &= I_{cm} \alpha_{cm} \\ R F_f &= \frac{2}{5} mR^2 \alpha_{cm} \\ F_f &= \frac{2}{5} mR \alpha_{cm} \end{aligned}$$

It is at this point that we observe the first complication; we do not know F_f , nor do we have a convenient substitution for it. We thus resort to Newton's second law, applied to the center of mass particle, in the y dimension along the incline, for the required substitution.

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_{cm} \\ mg \sin \theta - F_f &= ma_{cm} \\ F_f &= m(g \sin \theta - a_{cm}) \end{aligned}$$

We equate these two results,

$$m(g \sin \theta - a_{cm}) = \frac{2}{5} m R \alpha_{cm}$$

A few steps of algebra yield,

$$g \sin \theta = \frac{2}{5} R \alpha_{cm} + a_{cm}$$

To complete the derivation, we recall that the angular acceleration about the center of mass axis is equivalent to the angular acceleration about any other parallel axis. We choose the z axis at the point of contact, $\alpha_{cm} \equiv \alpha_z$, and substitute,

$$g \sin \theta = \frac{2}{5} R \alpha_z + a_{cm}$$

As seen in solution 1, we now identify $R \alpha_z$ as a_{cm} based on the third circular to angular transformation and substitute.

$$g \sin \theta = \frac{2}{5} a_{cm} + a_{cm} = \frac{7}{5} a_{cm}$$

We solve for a_{cm} .

$$a_{cm} = \frac{5}{7} g \sin \theta$$

The result is of course identical. The angular acceleration is most easily found by applying the third circular to angular transform, $\alpha = \frac{a_{cm}}{R}$.

Specific values for the posed problem give:

$$a_{cm} = \frac{5}{7} (9.8 \text{ m/s}^2) \sin 10^\circ = 1.21 \text{ m/s}^2$$

$$\alpha_{cm} = \frac{1.21 \text{ m/s}^2}{.4 \text{ m}} = 3.04 \text{ rad/s}^2$$

Finding V_{cm} , solution 1. In this approach, we simply use the linear motion equation, $v_f^2 - v_i^2 = 2 \vec{a} \cdot \Delta \vec{s}$, and the acceleration obtained from the previous solution. Here, $v_i=0$, and $\Delta s = \Delta h / \sin \theta$.

$$v_{cm,f}^2 = 2 a_{cm} \Delta s$$

$$v_{cm,f} = \sqrt{2 a_{cm} \Delta s}$$

$$v_{cm,f} = \sqrt{2(1.21 \text{ m/s}^2) \left(\frac{2 \text{ m}}{\sin 10^\circ} \right)} = 5.23 \text{ m/s}$$

This is the quickest way to solve for v_{cm} if a_{cm} is known.

Finding v_{cm} , solution 2. In this approach, we use the work energy theorem. We select the earth frame of reference; the z axis origin is placed

at the point of contact of the sphere. The version of the work energy theorem we will use is:

$$Work_{\text{all except mg}} = \Delta KE + \Delta GPE_{cm}$$

Reviewing the work done, C does no work since the angle between C and the displacement is 90 degrees; F_f does no work since the displacement is zero (no slippage), and the work done by mg is excluded. Thus, no work is done.

We will treat the kinetic energy change as purely rotational about the z contact axis. The gravitational potential energy change will be negative (we control the sign explicitly).

$$0 = \frac{1}{2} I_z \Delta(\omega_z^2) - |mg\Delta h|$$

As in the previous examples, I_z is computed using the parallel axis theorem as $I_z = 7/5 mR^2$, and upon substitution we have,

$$0 = \frac{1}{2} \left(\frac{7}{5} mR^2 \right) \omega_{z,f}^2 - |mg\Delta h|$$

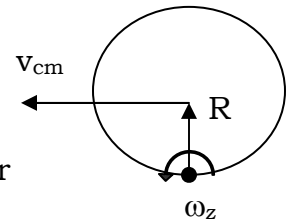
a few steps of algebra gives,

$$|g\Delta h| = \frac{7}{10} R^2 \omega_z^2$$

We identify $R^2 \omega_{z,f}^2 = v_{cm,f}^2$ using the second circular to angular transformation and replace in the above equation to give,

$$v_{cm,f} = \sqrt{\frac{10}{7} g\Delta h} = 5.23 \text{ m/s}$$

The value obtained is identical to solution 1.



Finding v_{cm} , solution 3. This approach is quite similar to solution 2; we use the work energy theorem. We select the earth frame of reference; the z axis origin is placed at the point of contact of the sphere. The version of the work energy theorem we will use is:

$$Work_{\text{all except mg}} = \Delta KE + \Delta GPE_{cm}$$

Reviewing the work done, C does no work since the angle between C and the displacement is 90 degrees; F_f does no work since the displacement is zero (no slippage), and the work done by mg is excluded. Thus, no work is done. The work analysis is identical to that in solution 2.

What is different is that we will treat the kinetic energy change as a combination of the kinetic energy of the center of mass particle and add the rotational energy about the center of mass axis. The gravitational potential energy change will be negative (we control the sign explicitly).

$$Work_{\text{all except mg}} = \Delta KE_{cm} + \Delta RE_{cm} + \Delta GPE_{cm}$$

$$0 = \frac{1}{2} m v_{cm,f}^2 + \frac{1}{2} I_{cm} \omega_{cm}^2 - |mg\Delta h|$$

$$|mg\Delta h| = \frac{1}{2}mv_{cm,f}^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega_{cm}^2$$

$$|mg\Delta h| = \frac{1}{2}mv_{cm,f}^2 + \frac{2}{10}mR^2\omega_{cm}^2$$

At this point, we recall $\omega_{cm} = \omega_z$, and $R^2\omega_z^2 = v_{cm}^2$. Substitution in the above gives,

$$|mg\Delta h| = \frac{1}{2}mv_{cm,f}^2 + \frac{2}{10}mv_{cm,f}^2$$

$$|mg\Delta h| = \frac{7}{10}mv_{cm,f}^2$$

and solving for v_{cm} , we arrive at our result,

$$v_{cm,f} = \sqrt{\frac{10}{7}g\Delta h}$$

which is identical to solution 1 and 2.

The combination approach using the kinetic energy of the center of mass plus the rotational energy about the cm axis often seems to be the easiest for beginning students since it simply adds the change in rotational energy term to the already familiar form of the work energy theorem.

Finding v_{cm} , solution 4. This approach is by far the least efficient solution. It is presented here only to illustrate how the frame of reference effects the work energy distribution.

We apply the work energy theorem of form, $Work_{\text{all except mg}} = \Delta KE + \Delta GPE_{cm}$. What is different is that we will select as a frame of reference, the center of mass of our sphere. In this frame of reference, we observe neither a change in altitude, nor a change in center of mass velocity; only a change in the micro kinetic energy of rotation.

We will use as the form of the work calculation, $Work = \int \vec{\tau} \cdot d\vec{\theta}$ which reduces to $|\tau||\Delta\theta|$ for a constant torque. The work audit suggests that the forces C and mg produce no torque about the center of mass axis (note mg would be excluded anyway do to the presence of the change in gravitational potential energy) and thus does no work. The F_f is another matter. In this frame of reference, the force of friction produces a constant torque about the center of mass axis which results in an angular displacement. Work is done by F_f . Our work energy equation becomes,

$$\tau_{F_f} \Delta\theta = \Delta RE_{cm}$$

$$R F_f \Delta\theta = \frac{1}{2}I_{cm}\omega_{cm,f}^2$$

We substitute for the moment of inertia and reduce,

$$R F_f \Delta\theta = \frac{1}{2} \left(\frac{2}{5} m R^2 \right) \omega_{cm,f}^2$$

$$R F_f \Delta\theta = \frac{2}{10} m R^2 \omega_{cm,f}^2$$

Note that from the first circular to angular transform, $R\Delta\theta = \Delta s$.

$$F_f \Delta s = \frac{2}{10} m R^2 \omega_{cm,f}^2$$

Here again, we run into the problem of a suitable replacement for F_f . We utilize Newton's second law applied to the center of mass particle in the y dimension along the incline; $mg \sin \theta - F_f = ma_{cm}$.

We solve for F_f to give $F_f = m(g \sin \theta - a_{cm})$ and replace the force of friction in the above equation,

$$m(g \sin \theta - a_{cm}) \Delta s = \frac{2}{10} m R^2 \omega_{cm,f}^2$$

$$g \Delta s \sin \theta - a_{cm} \Delta s = \frac{2}{10} R^2 \omega_{cm,f}^2$$

Let us recognize what we have above; $\Delta s \sin \theta \equiv \Delta h$, $\omega_{cm,f} \equiv \omega_{z,f}$, and

$R^2 \omega_{z,f}^2 \equiv v_{cm}^2$. Substitution gives,

$$g \Delta h - a_{cm} \Delta s = \frac{2}{10} v_{cm,f}^2$$

This now leaves us with the task of eliminating a_{cm} . Recall from linear motion, $v_f^2 - v_i^2 = 2 \vec{a}_{cm} \bullet \Delta \vec{s}$. Setting $v_i = 0$ and rearranging we get

$a_{cm} \Delta s = \frac{1}{2} v_{cm,f}^2$ which we will substitute.

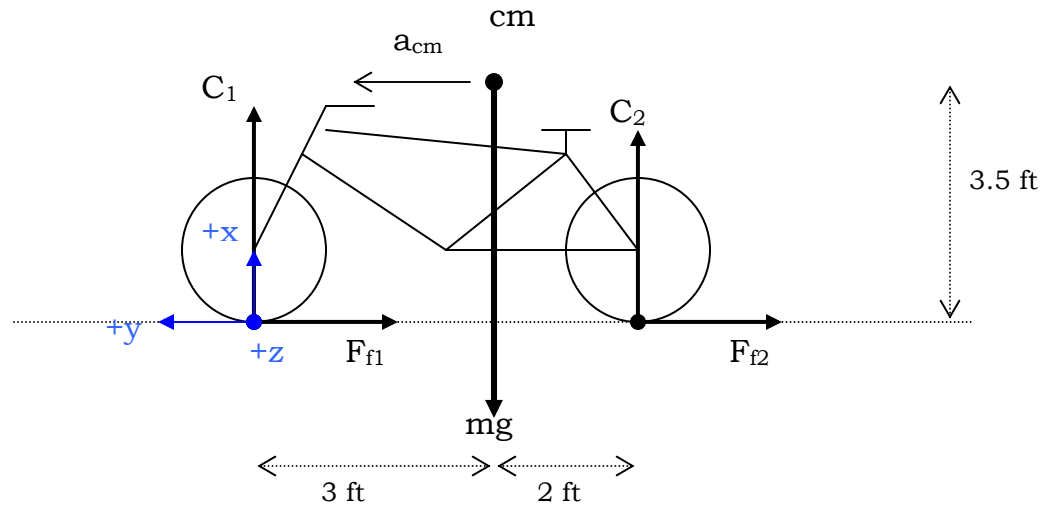
$$g \Delta h - \frac{1}{2} v_{cm,f}^2 = \frac{2}{10} v_{cm,f}^2$$

Final rearrangement produces the desired and identical result,

$$v_{cm,f} = \sqrt{\frac{10}{7} g \Delta h}$$

Clearly a "scenic" exercise. The journey does however illustrate the fact that the work and energy quantities may distribute themselves in different forms and places when viewed from different frames of reference.

Example. A 200 lb bicycle (and rider) suddenly applies both brakes to stop. The bicycle slides along the road way. Coefficient of friction between the tires and road is .6 . We wish to determine the center of mass acceleration and the friction forces on each tire. The geometry of the bicycle and rider, and forces involved, are shown in the figure below.



We choose the translational coordinate system as shown, the location of the z axis origin will be at the contact point of the front tire. With this choice, we recall there is a non-zero angular acceleration (see Rotational Equilibrium). The advantage of this selection is that many of the forces involved produce zero torque about the axis.

We perform a linear force analysis to obtain a_{cm} . The system equations are:

System : m_{total} , y dim(hor), a_{cm} $-F_{f1} - F_{f2} = m_t a_{cm}$	System : m_{total} , x dim(vert), $a = 0$ $C_1 + C_2 = mg$	Friction model : $F_{f1} = \mu C_1, F_{f2} = \mu C_2$
--	---	--

The friction model equations are substituted into the y dim equation to give,

$$-\mu C_1 - \mu C_2 = m a_{cm}$$

The equation is factored,

$$-\mu(C_1 + C_2) = m a_{cm}$$

The x dim equation is substituted for the sum of the compression forces,

$$-\mu mg = m a_{cm}$$

The mass is canceled and we have our result.

$$a_{cm} = -\mu g = -(.6)(32 \text{ f/s}^2) = -19.2 \text{ f/s}^2$$

We now address the issue of torques. The equation is:

$$\sum \vec{\tau}_{ex} = m_{total} (\vec{R}_{cm} \times \vec{a}_{cm})$$

With the chosen axis, only mg and C_2 produce torques. Our equation is:

$$(5 \text{ ft}) C_2 - (3 \text{ ft}) mg = m(3.5 \text{ ft } a_{cm})$$

Using $mg=200$ lbs, $m=6.25$ slugs, and $a_{cm}=-19.2$ f/s², we solve for C_2 to get
 $C_2=36$ lbs

From the vertical force balance, $C_1=164$ lbs. The friction forces may now be obtained from the friction model equations as

$$F_{f1}=98.4 \text{ lbs and } F_{f2}=21.6 \text{ lbs}$$

or a total friction force of 120 lbs.

As a quick check, we evaluate $|ma_{cm}| = 6.25 \text{ slugs } 19.2 \text{ f/s}^2 = 120 \text{ lbs!}$

We will leave it to the reader to verify that if the z axis were placed at the center of mass, the sum of the torques would equal zero about this axis, and that analysis would yield the same result.

17. Appendix I, Moment of Inertia Computations. Here we derive the specific formula for the moment of inertia of selected objects about various axis through implementation of the definition,

$$I_z = \int R_{\perp z}^2 dm$$

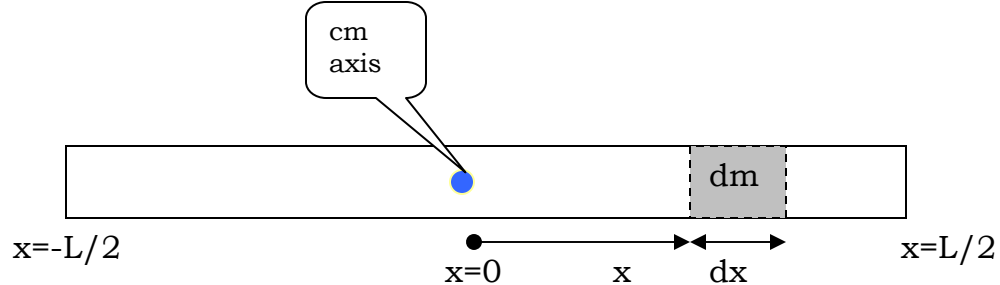
Before the student of physics reviews these derivations however, a few common thoughts must be emphasized:

- We draw attention to the coordinate system, and its origin. When we select the axis of revolution, for example the z axis, any element of matter (dm) that lies on that axis ($x=y=0$) must contribute zero moment of inertia to the object. Its R_z must be zero. Where you put your origin is extremely important.
- An often overlooked fact is that the coordinate axis has both a plus and minus direction. Recall we are using the coordinate system to “target” (address) elements of matter.
- Consideration of the objects symmetry. If the object possesses symmetry about the chosen axis, it will be more efficient if we take advantage of that symmetry.
- The variable dm in the moment of inertial definition has important *physical* significance but is useless in the evaluation of the integral itself. We will consistently use the fact that the object is homogeneous and uniform, and thus of constant density (ρ). Therefore, $\rho = \frac{M_{total}}{Volume_{total}} = \frac{dm}{dV}$ and $dm = \rho dV$ may be used to replace dm in the integral.
- Finally, our tactic will in general be to:
 1. Sketch the object.
 2. Identify the axis.
 3. Identify the differential element taking advantage of symmetry.
 4. Target the element using the coordinate system.
 5. Identify the limits of the integral.
 6. Perform the integral.

We proceed.

Long, infinitely thin object, cm axis.

By definition, infinitely thin implies that the cross section of the object has no distinguishing shape; the object can be considered a long thin round rod, a long thin square rod, etc. Also, infinitely thin means that we need only consider the objects length in computing the moment of inertia.



Above, we have our sketch. The axis is identified as out of the page and the origin is at the center of mass. The differential element is located at x and its length is dx . The limits are identified. We present the integral:

$$I_{cm} = \int_{-L/2}^{+L/2} x^2 dm$$

To replace dm , we observe the one dimensional nature of this integral and will thus use the linear mass density: the mass per unit length.

$\rho_l = \frac{m}{L} = \frac{dm}{dx}$ and so the substitution becomes $dm = \rho_l dx$,

$$I_{cm} = \rho_l \int_{-L/2}^{+L/2} x^2 dx$$

The integral is executed to give,

$$I_{cm} = \rho_l \frac{1}{3} (x^3)_{-L/2}^{+L/2}$$

and evaluated,

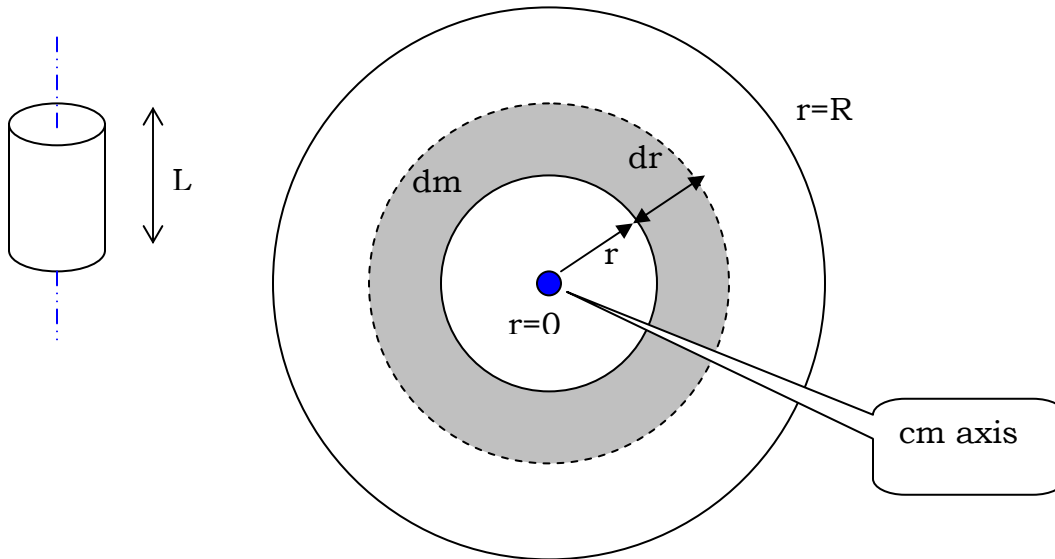
$$I_{cm} = \rho_l \frac{1}{3} \left(\frac{L^3}{8} - \frac{(-L)^3}{8} \right)$$

$$I_{cm} = \frac{1}{12} \rho_l L^3$$

At this point, we replace the linear mass density with $\rho_l = m/L$, and arrive at our final form,

$$I_{cm} = \frac{1}{12} mL^2$$

Circular Cylinder of length L, axis along length, through cm. Below is our sketch, top view, with steps 1 – 4 complete. The axis is out of the page and through the center of mass. We will take advantage of the circular symmetry by selecting the differential element as a ring of thickness dr and length L . The limits of integration will be from $r=0$ to $r=R$. Since the object has area, we will use the area mass density; the mass per unit area.



We start with our moment of inertia definition,

$$I_{cm} = \int_0^R r^2 dm$$

To replace dm , we utilize the mass per unit volume; $\rho = \frac{m}{V} = \frac{dm}{dV}$ and solving for dm we get, $dm = \rho dV$.

The volume of our differential ring, from geometry, is, $dV = (2\pi r)dr L$. Thus $dm = \rho L(2\pi r)dr$ and we substitute into the integral.

$$I_{cm} = L\rho 2\pi \int_0^R r^3 dr$$

The integral is executed and evaluated to give,

$$I_{cm} = \frac{2}{4} L\rho \pi (r^4)_0^R = \frac{1}{2} L\rho \pi R^4$$

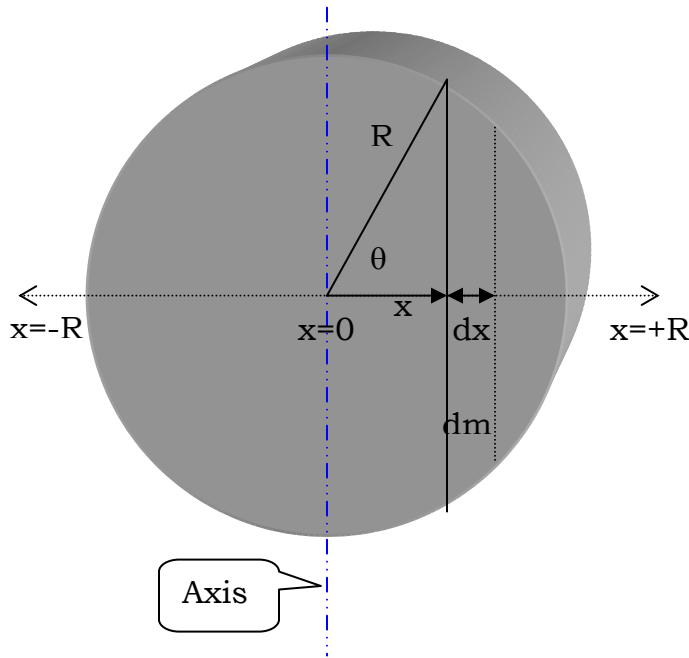
We replace the density, $\rho = \frac{m}{\pi R^2 L}$ to get,

$$I_{cm} = \frac{1}{2} mR^2$$

Notice that the length of the cylinder does not appear in the final result.

Flat Circular Plate, infinitely thin, axis within the plane of the area.

This object is somewhat different than the long thin rod in that it has a finite area and is thus two dimensional. We will take this opportunity to illustrate a trigonometric substitution of variable. Below is our sketch with steps 1 through 4 complete. The axis is within the page as shown and through the center of mass. A vertical strip will be chosen as the differential element. The perpendicular distance to the differential element is x .



The integral to evaluate is: $I_{cm} = \int_{-R}^{+R} x^2 dm$. To put this in executable form we require a few substitutions,

$$\rho_A = \frac{m}{A} = \frac{dm}{dA} \text{ thus, } dm = \rho_A dA$$

and

$$dA = 2(R \sin \theta) dx, \text{ from geometry}$$

Since θ is a function of x , we will replace x with θ and take the integration limits from $\theta = \pi$ (formerly $-R$) to $\theta = 0$ rad (formerly $+R$). From trigonometry,

$$x = R \cos \theta$$

and differentiation of the above gives,

$$dx = -R \sin \theta d\theta$$

Replacement of dm with much substitution gives,

$$dm = \rho_A dA = \rho_A 2R \sin \theta dx = \rho_A 2R \sin \theta (-R \sin \theta) d\theta = -2\rho_A R^2 (\sin \theta)^2 d\theta$$

Substitution of x and dm into the integral gives,

$$I_{cm} = \int_{-R}^R x^2 dm = 2\rho_A \int_0^\pi (R \cos \theta)^2 (R \sin \theta)^2 d\theta$$

Now from trigonometry $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$. We substitute and simplify,

$$I_{cm} = \frac{1}{2} \rho_A R^4 \int_0^\pi (\sin 2\theta)^2 d\theta$$

From tables of integrals, we find $\int (\sin ax)^2 dx = \frac{1}{2}x - \frac{1}{4a} \sin(2ax)$ and apply it to our integral,

$$I_{cm} = \frac{1}{2} \rho_A R^4 \int_0^\pi (\sin 2\theta)^2 d\theta = \frac{1}{2} \rho_A R^4 \left\{ \frac{\theta}{2} - \frac{1}{4} \sin(4\theta) \right\}_0^\pi$$

which simplifies to,

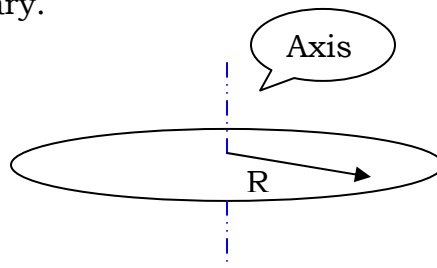
$$I_{cm} = \frac{\pi}{4} \rho_A R^4$$

Replacing the density with $\rho_A = \frac{m}{\pi R^2}$, we arrive at the final result,

$$I_{cm} = \frac{1}{4} m R^2$$

Circular Hoop, Infinitely Thin, Axis normal to Hoop through cm.

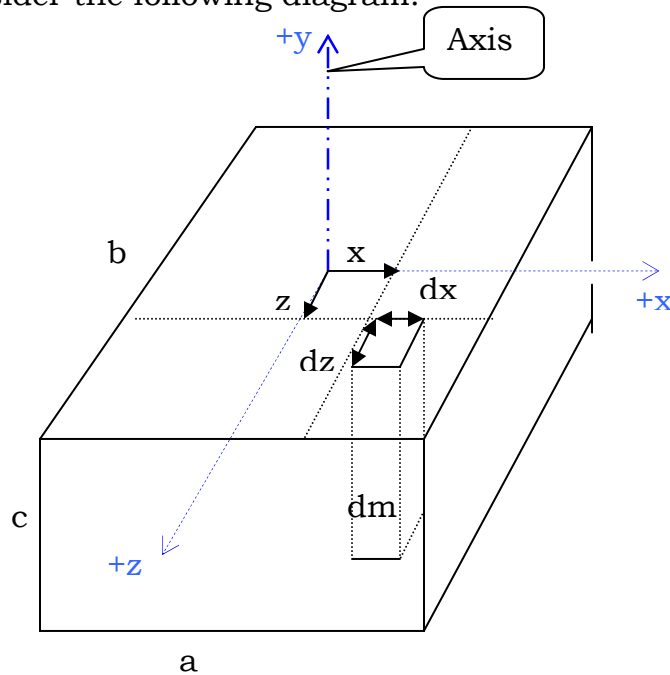
Since all the mass of the hoop is at a perpendicular distance R from its center, the moment of inertia formula is identical to that of a point mass. No derivation is necessary.



$$I_{cm} = mR^2$$

Rectangular Slab, cm Axis.

We take this opportunity to introduce an interesting and efficient technique called a multiple integral to compute the moment of inertia of a rectangular slab. The multiple integral is often used to address points in space on an object when the object and the coordinate system have similar symmetry. The technique is based on the fact that the vector components are independent of each other (orthogonal) and may therefore be changed (moved) independently of each other to address (target) each point in the object. Consider the following diagram.



Our differential element, dm , has a volume $dV = c \, dx \, dz$. We will use the volumetric mass density, $\rho = \frac{m}{V} = \frac{dm}{dV}$, and solving for dm , $dm = \rho \, dV = \rho \, c \, dx \, dz$.

Our integral for consideration is,

$$I_{cm} = \int R_{\perp}^2 dm$$

In the above integral, $R_{\perp}^2 = x^2 + z^2$ and so we replace it.

$$I_{cm} = \int (x^2 + z^2) dm$$

As you are observing, we are systematically replacing our variables with functions of x and z . Once this is accomplished, the double integral will effectively move our differential element over the $x - z$ surface, collecting each element's contribution to the moment of inertia of the slab.

Next, we replace dm and show the result as a double integral. Notice the limits of integration will sweep the $x - z$ surface of our slab.

$$I_{cm} = \int_{-b/2}^{+b/2} \int_{-a/2}^{+a/2} (x^2 + z^2) \rho c \, dx \, dz$$

At this point, we explain the procedure to evaluate the above double integral. In our case, we have two variables, x , and z . We will start with x and integrate considering z to be constant. Then we will integrate again on the remaining variable, z . The order should not be important since x and z are orthogonal. This method in effect, sums the x components first, and then the z components.

Starting with,

$$I_{cm} = \int_{-b/2}^{+b/2} \int_{-a/2}^{+a/2} (x^2 + z^2) \rho c \, dx \, dz$$

First we integrate on x ,

$$I_{cm} = \rho c \int_{-b/2}^{+b/2} \left[\frac{x^3}{3} + z^2 x \right]_{-a/2}^{+a/2} dz$$

and evaluate.

$$I_{cm} = \rho c \int_{-b/2}^{+b/2} \left[\frac{a^3}{12} + z^2 a \right] dz$$

Next, we integrate on z (a is constant),

$$I_{cm} = \rho c \left[\frac{a^3}{12} z + a \frac{z^3}{3} \right]_{-b/2}^{+b/2}$$

and evaluate.

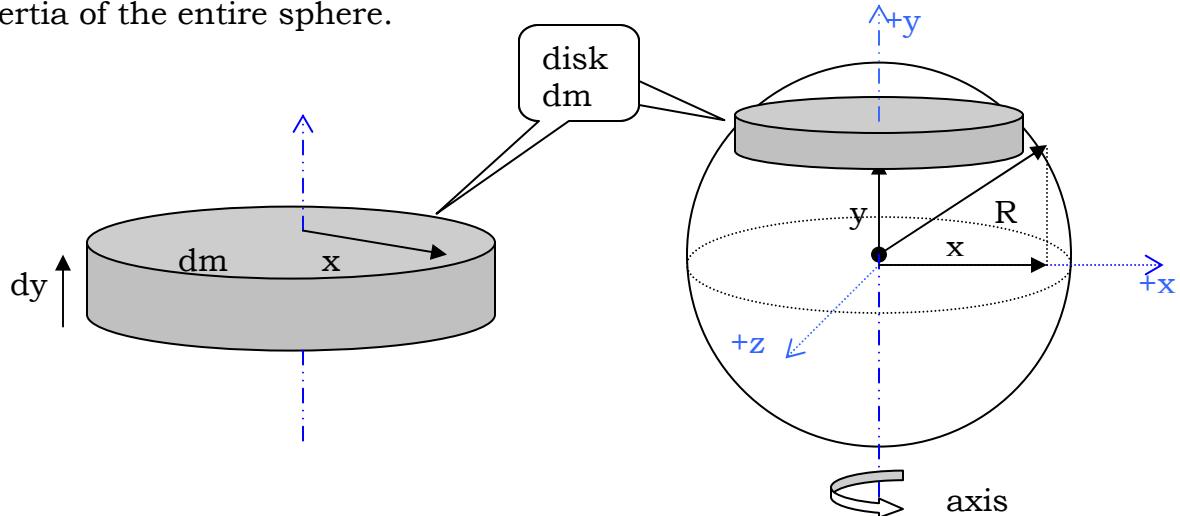
$$I_{cm} = \frac{1}{12} \rho c (a^3 b + ab^3)$$

Finally, we replace the density with $\rho = \frac{m}{abc}$, and arrive at our result:

$$I_{cm} = \frac{1}{12} m (a^2 + b^2)$$

Note that the dimension, c , does not appear in the final result. This is as expected since that dimension is parallel the axis of the calculation.

Solid Sphere, cm axis. Rather than using the definition, $I = \int R^2 dm$, to compute the moment of inertia, our tactic will be to select a differential element in the shape of a thin circular disk; a slice of the sphere. We then sum (through integration) the contribution of each slice to the moment of inertia of the entire sphere.



We start with our summation equation,

$$I_{cm} = \int_{all} dI$$

Borrowing the result for the moment of inertia of a cylinder, our disk is infinitely thin and of infinitely small mass, and as such has an infinitely small moment of inertia, $dI_{cm} = \frac{1}{2} dm x^2$. We will require a replacement for both dm and x^2 .

For the dm substitution, we replace as $dm = \rho dV = \rho \pi x^2 dy$. For the x substitution, we rely on the Pythagorean theorem, $R^2 = x^2 + y^2$, thus $x^2 = R^2 - y^2$. We now replace.

$$I_{cm} = \int_{all} dI = \int \frac{1}{2} x^2 dm = \int \frac{1}{2} (R^2 - y^2) \rho \pi (R^2 - y^2) dy$$

$$I_{cm} = \frac{1}{2} \rho \pi \int_{-R}^{+R} (R^2 - y^2)^2 dy$$

We expand,

$$I_{cm} = \frac{1}{2} \rho \pi \int_{-R}^{+R} (R^4 - 2R^2 y^2 + y^4) dy$$

and change the limits of integration,

$$I_{cm} = 2 \frac{1}{2} \rho \pi \int_0^R (R^4 - 2R^2 y^2 + y^4) dy$$

and perform the integration and evaluation,

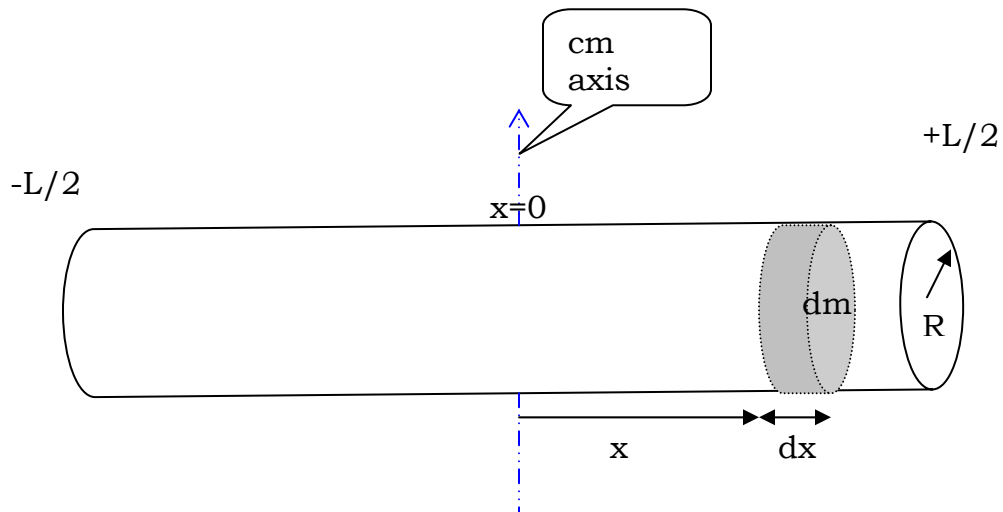
$$I_{cm} = \rho \pi \left[R^5 - \frac{2}{3} R^5 + \frac{1}{5} R^5 \right]$$

$$I_{cm} = \frac{8}{15} \rho \pi R^5$$

and finally, we replace the density with $\rho = \frac{m}{\frac{4}{3} \pi R^3}$, to arrive at our result,

$$I_{cm} = \frac{2}{5} m R^2$$

Long circular rod, cm axis, normal to the length. In this final calculation, we will use yet another method. We will incorporate the result of the “Flat Circular Plate, infinitely thin, axis within the plane of the area” derivation outcome, and the parallel axis theorem. The tactic is to first select the flat circular plate as our differential element, to offset the axis of our flat circular plate using the parallel axis theorem, and to sum, using integration, the contribution of each plate in the circular rod.



We start with the summation, $I_{cm} = \int_{all} dI$. The moment of inertia of the

infinitely thin disk about its own axis, as derived, is $dI = \frac{1}{4} dm R^2$.

Using the parallel axis theorem, we adjust the moment of inertia to the Rod's cm axis as,

$$dI = \frac{1}{4} dmR^2 + dm x^2$$

and replace it in the integral.

$$I_{cm} = \int \left(\frac{1}{4} R^2 dm + x^2 dm \right)$$

We now seek a substitution for dm using the density. $\rho = \frac{m}{V} = \frac{dm}{\pi R^2 dx}$ and so $dm = \rho \pi R^2 dx$. We substitute to get,

$$I_{cm} = \int \left(\frac{1}{4} R^2 + x^2 \right) \rho \pi R^2 dx$$

Removal of the constants from the integral gives,

$$I_{cm} = \rho \pi R^2 \int_{-L/2}^{+L/2} \left(\frac{1}{4} R^2 + x^2 \right) dx$$

and a limit change for ease of evaluation gives,

$$I_{cm} = 2 \rho \pi R^2 \int_0^{+L/2} \left(\frac{1}{4} R^2 + x^2 \right) dx$$

We integrate to give,

$$I_{cm} = 2 \rho \pi R^2 \left[\frac{1}{4} R^2 x + \frac{x^3}{3} \right]_0^{+L/2}$$

Evaluation of limits gives,

$$I_{cm} = 2 \rho \pi R^2 \left[\frac{R^2 L}{8} + \frac{L^3}{24} \right]$$

We now replace the density as $\rho = \frac{m}{\pi R^2 L}$, and arrive at our final form,

$$I_{cm} = \frac{1}{4} mR^2 + \frac{1}{12} mL^2$$

In comparing this result to the infinitely thin object, we see that the above formula carries an extra term for the finite round rod. It may also be interesting to note that for a rod with a length to diameter ratio of 10:1, the additional term affects the result at its third significant figure.

18. Appendix II. The Combination Method for determining Total Kinetic Energy.

The total energy of motion of a collection of particles is simply the scalar sum of the individual kinetic energies of all of the particles in the collection.

$$KE_{Total} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \dots$$

Since this total energy is based on each particles velocity, which is in turn based of some given frame of reference, the total energy must also be based on the frame of reference from which the velocities are measured.

Here, we wish to consider the kinetic energy of a group of particles as determined both from some fixed frame of reference (such as the earth frame of reference), and from the frame of reference of the collections center of mass, which is moving with respect to the fixed earth frame of reference.

Let v_{cm} be a particles velocity measured in the frame of reference of the collections center of mass, and let v_e be the particles velocity measured in the fixed earth frame of reference, and let V_{cm_e} be the velocity of the center of mass as measured in the fixed earth frame of reference.

The Galilean transformation relating these three velocities is:

$$v_e = v_{cm} + V_{cm_e}$$

The Total kinetic energy of the collection, as determined from the earth frame of reference, is thus

$$\left(KE_e \right)_{total} = \sum_{i=1}^n \frac{1}{2} m_i (v_e)_i^2$$

on application of the Galilean transformation,

$$\left(KE_e \right)_{total} = \sum_{i=1}^n \frac{1}{2} m_i (v_{cm} + V_{cm_e})_i^2$$

next, we expand the squared quantity,

$$\left(KE_e \right)_{total} = \sum_{i=1}^n \frac{1}{2} m_i (v_{cm}^2 + 2 v_{cm} V_{cm_e} + V_{cm_e}^2)_i$$

we distribute the summation,

$$\left(KE \right)_e^{total} = \sum_{i=1}^n \frac{1}{2} m_i (v_i)_{cm}^2 + V_{cm} \sum_{i=1}^n m_i v_i + \frac{1}{2} V_{cm} \sum_{i=1}^n m_i$$

we identify the first and third summation,

$$\left(KE \right)_e^{total} = \left(KE \right)_{cm}^{total} + V_{cm} \sum_{i=1}^n m_i (v)_{cm} + \frac{1}{2} M_{total} (V_{cm})^2$$

Since $M_{total} V_{cm} \equiv \sum_{i=1}^n m_i v_i$ from the definition of the center of mass velocity, the center summation becomes,

$$\left(KE \right)_e^{total} = \left(KE \right)_{cm}^{total} + V_{cm} \left(M_{total} V_{cm} \right) + \frac{1}{2} M_{total} (V_{cm})^2$$

and the velocity of the center of mass of the collection, in it's own frame of reference (the frame of reference of the center of mass) is equal to ZERO.

$$\left(KE \right)_e^{total} = \left(KE \right)_{cm}^{total} + \frac{1}{2} M_{total} (V_{cm})^2$$

Thus, the total kinetic energy of the system of particles in the earth frame of reference is equal to the total kinetic energy of the system of particles as determined from the frame of reference of the center of mass, PLUS the kinetic energy of the “center of mass particle” in the earth frame of reference.

For the case of a rigid solid which is rolling on a surface in the earth frame of reference, we may compute the total kinetic energy in the earth frame of reference by adding the “rotational energy” , computed from

$$RE_{cm} = \frac{1}{2} I_{cm} \omega^2 \text{ (center of mass frame of reference), to the translation kinetic}$$

energy of the “center of mass particle”, computed from $KE_e = \frac{1}{2} M_{total} v_{cm}^2$ in the earth frame of reference. This is the “Combination Method”.