

# Collections of Particles

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## Newton's Second Law.

**1. Introduction.** Newton's' second law is customarily presented to beginning students of physics as

$$\sum \vec{F}_{\text{ex}} = m \vec{a}$$

where the left side of the equation represents the sum of all the external force vectors and on the right side, m represents the system's mass (resistance to being accelerated), and "a" is the resulting vector acceleration of the system. Although the above relationship is true, it is not what Newton said.

Newton expressed his second law in a much more general statement. The second law of Newton is:

$$\sum \vec{F}_{\text{ex}} = \frac{d(m\vec{v})}{dt}$$

In words, the sum of the external force vectors equals the time rate of change of momentum; the mv product representing momentum.

One can deduce  $F = ma$  from Newton's second law by considering the mass to be constant, and as such, the mass can be removed from the differential and the resulting derivative,  $dv/dt$ , interpreted as acceleration.

$$\sum \vec{F}_{\text{ex}} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} = m\vec{a}$$

One may argue that the two statements above are for all purposes equivalent however the very fact that the mass of the system does not need to be constant lends generality and power to Newton's law.

There are at least two interesting variations of Newton's law which lend a systematic approach to the solution of many problems in engineering. They are both derived from the 2<sup>nd</sup> law above. The first variation is called Impulse – Momentum and the second is called The Flow  $F=ma$ . We shall derive each variation.

## 2. Impulse – Momentum.

Starting with the second law above,

$\sum \vec{F}_{ex} = \frac{d(m\vec{v})}{dt}$ , we multiply both sides of the equation by dt.

$$\sum \vec{F}_{ex} dt = d(m\vec{v})$$

Next, we select a time interval so that we may integrate both sides of the equation.

$$\int (\sum \vec{F}_{ex}) dt = \int d(m\vec{v})$$

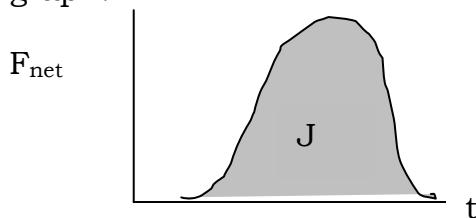
We apply the integral on the right side of the equality to obtain the Impulse – Momentum equation:

$$\int (\sum \vec{F}_{ex}) dt = \Delta(m\vec{v})$$

**3. Impulse.** The expression on the left side of the equality is called the **Impulse**. One may notice a similarity to the definition of work. There is however a major difference. The Impulse is a **vector time path function**. To evaluate the Impulse, one must know how the external vector forces vary as a function of time along the time path from initial to final time. Typically, the letter “J” is used as a shorthand for this integral.

$$\vec{J} \equiv \int (\sum \vec{F}_{ex}) dt$$

Similar to work, the impulse may be viewed as the area under a Fnet vs. time graph.



Evaluating an integral of this nature using graphical methods however must be done with the understanding that Impulse, and Force, are vector quantities.

**4. Momentum Change.** The expression on the right side of the equality is the **Change in Momentum**. One may again notice a similarity to the kinetic energy concept. The change in Momentum is a Point Function. There is however a major difference. The Change in Momentum is a **vector Point Function**, and it does not require the mass to be constant! This, *is discovery!* The symbol “ $\Delta p$ ” is often used as shorthand for change in momentum, we will not here.

The Impulse momentum equation may now be written as:

$$\vec{J} = \Delta(m\vec{v})$$

**5. General Application.** Impulse – Momentum is rarely applied to a single particle although it is perfectly acceptable to do so. Systems containing Multiple Particles are far better candidates for Impulse – Momentum. Here, the power of Impulse – Momentum is striking.

Consider two particles which are interacting in a complex way. We apply Impulse – Momentum to each individual particle:

$$\vec{J}_1 = \int \vec{F}_1 dt = \Delta(m\vec{v})_1 \quad \text{and} \quad \vec{J}_2 = \int \vec{F}_2 dt = \Delta(m\vec{v})_2$$

We now add the two equations.

$$\int (\vec{F}_1 + \vec{F}_2) dt = \Delta(m\vec{v})_1 + \Delta(m\vec{v})_2$$

In doing so, all forces involved in the interaction between the two particles will add to zero as stated in Newton's 3<sup>rd</sup> law. These forces are now Internal to the system of both particles. The resulting Impulse will contain only forces which are External to this collective system. For a collective system, Impulse – Momentum becomes:

$$\vec{J}_{\substack{\text{external} \\ \text{to System}}} = \sum_{i=1}^n \Delta(m\vec{v})_i$$

We may entirely eliminate all complex interaction forces from the Impulse side of the equation, leaving only forces which are external to the collective system of particles. It is in this fact where the true power of the Impulse – Momentum equation lies. In some cases, the Impulse could result in a value of zero; and we would say that "*Momentum is Conserved*".

Momentum is not in general conserved for systems smaller than the universe. Engineering problems rely on systems much smaller than the universe and as such require the engineer and/or scientist to closely examine the force condition for any external force which can act for any time duration at all between initial and final. The force does not have to displace the object (such as in Work), merely act on it.

Finally, this is a **vector** relationship. To the student of physics this implication should be clear. It is identical to the approach when using  $F=ma$ . In a multiple dimensional application, the student will:

- Select a collection of particles for the System.
- Assign a coordinate system with + and – clearly noted.

- Provide a force vector diagram showing all external forces which act on any particles within the system.
- Replace the force vectors with component pairs directed within the coordinate space.
- Author an Impulse – Momentum equation for each dimension using the appropriate components of both force and velocity.
- Solve the equations for what ever relationship can be obtained.

## 6. Example in Application.

### Problem.

A rocket sled of having a mass of 3000 kg is moving at a velocity of 290 m/s on a set of rails when its engine burns out. There is little friction between the sled and rails. At that point, a scoop drops from the sled into a trough of water between the rails and scoops up 950 kg of water into an empty tank on the sled. What will be the velocity of the sled after the water is elevated into the rocket sled's tank?

### Solution.

We may be tempted to consider a Work – Energy solution to this problem. This attempt will be short lived however as we discover that there is much work done in the collision between the sled and water. This collision is not elastic; the water will gain considerable thermal energy and its temperature will rise as a result. We now consider Impulse – Momentum.

The situation is in essence a two body horizontal inelastic collision. We select as the system the sled and the water. We choose positive in the direction of the sled's velocity. We consider only the horizontal dimension in the vector analysis. For the time audit, we select *initial* at the time just prior to the scoop being deployed. *Final* is selected at the time all of the water is loaded into the sled and we seek to know its speed. We introduce the Impulse – Momentum equation.

$$\vec{J}_x = \Delta(m\vec{v}_x)_{sys}$$

Reviewing the Impulse, the only horizontal force involved is the force between the scoop and the water below. For the system of the rocket and water, this is an internal force and as such does not constitute an Impulse to the system. Thus, the systems horizontal impulse is ZERO and momentum is conserved.

Reviewing the system's change in Momentum, both the sled and the water will change momentum. We also note that the final velocity of the sled is identical to the final velocity of the water it collects and the initial velocity of the water is zero. We fabricate the Impulse Momentum equation and solve for  $v_f$  :

$$\begin{aligned}0 &= \Delta(m\vec{v})_{sled} + \Delta(m\vec{v})_{water} \\0 &= m_{sled}(v_f - v_{i\text{ sled}}) + m_{water}(v_f - v_{i\text{ water}}) \\(m_{sled} + m_{water})v_f &= m_{sled}v_{i\text{ sled}} \\v_f &= \frac{m_{sled}}{(m_{sled} + m_{water})}v_{i\text{ sled}} \\v_f &= \frac{3000 \text{ kg}}{(3000 + 950)\text{kg}}290 \text{ m/s} = 220.3 \text{ m/s}\end{aligned}$$

## Center of Mass

**1. Introduction.** Center of gravity, center of mass, this concept seems very familiar. Indeed, many people including students of science have used the phrase in daily conversation. Yet, “what is the center of mass?”, and of more concern, what are its properties? We explore these ideas in the attempt to gain clarity.

If we were to present the question, “What is the center of mass?”, our poll would probably result in one of the following answers: ‘It’s where all of the mass of the object is”. “It’s the balance place”.

To the response “where all the mass of an object is”, it is doubtful that anyone truly believes that all the mass of an object is concentrated in a pea size pellet, sitting inside a holographic projection otherwise known as the object. Yet the words, center of mass, do tend to interpret in that way, “mass at the center”. To the response “balance place”, we are in agreement that an object, if it will balance at all, will only do so at one place and that place is often thought of as the center of mass.

Perhaps we could say that the center of mass is similar to “Santa Clause”. We describe or define Santa Clause as a “jolly ole fellow wearing a red suit and most importantly, having a large bag of toys”. Santa has properties as well, “with the help of a finger placed at a coordinate near his nose, up the chimney he goes”. Yet, no one has ever seen Santa. With all condolences to my 23 and 24 year old daughters, Santa **does not exist!** Santa is fiction. I’m sure their response would be that “Santa does exist, and the fact that I have made the above cynical statement only shows that my shallow concept of existence, is based on the sense of sight, not faith”. Indeed, the subject of existence seems to transcend beyond that which is physical. Velocity is an idea and this idea exists because we define it to exist; the idea has useful properties. In this spirit, we now declare the existence of “Center of Mass”.

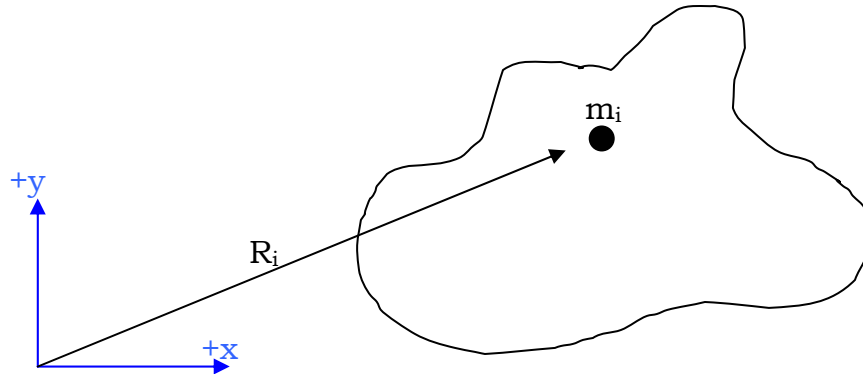
To develop the concept of center of mass, we will start with the pretense of the point mass. We then evolve the pretense to objects which have mass distributed in space. The remainder of this discussion will peruse the properties of our creation.

**2. Location.** The center of mass has a location. More formally, a coordinate in space. This coordinate is based on a pre selected three dimensional coordinate system. This makes the location of the center of mass a Location Vector. The location of the center of mass will be defined as the location weighted average of the masses of the individual particles which make up the collection of particles for which we seek the center of mass. This description is consistent with the notion of “balance point” in that it is effectively equivalent to the summation of the torques,

due to the weights of each of the particles involved, equal zero. For a two dimensional collection of masses,

$$M_{total} X_{cm} = m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots + m_n x_n$$

$$M_{total} Y_{cm} = m_1 y_1 + m_2 y_2 + m_3 y_3 + \dots + m_n y_n$$



The above equations can be written in a more compact form using vectors. The location vector for a collection of point masses is defined as:

$$M_{total} \vec{R}_{cm} = m_1 \vec{R}_1 + m_2 \vec{R}_2 + m_3 \vec{R}_3 + \dots + m_n \vec{R}_n$$

or

$$M_{total} \vec{R}_{cm} = \sum_{i=1}^n \vec{R}_i m_i$$

where,

$M_{total}$  is the total mass of the collection of point masses.

$R_i$  is the location of the  $i^{\text{th}}$  point mass.

$m_i$  is the mass of the  $i^{\text{th}}$  point mass.

$R_{cm}$  is the location vector to the center of mass of the collection.

This definition can be extended to an object of distributed mass by considering the object to be composed of an infinite collection of infinitely small point masses, located at various vector locations within the object. The above summation then reverts to an integration over the body of the object.

$$M_{total} \vec{R}_{cm} = \int_{\text{object}} \vec{R} dM$$

In practice, this integral is executed by replacement of  $dM$ , using density and a differential volume  $dV$ , as  $dM = \rho dV$ .

**3. Beyond Location.** Having established an equation to define the center of mass position vector, we extend our thoughts to differentiating this equation with respect to time assuming constant mass.



$$M_{total} \vec{R}_{cm} = m_1 \vec{R}_1 + m_2 \vec{R}_2 + m_3 \vec{R}_3 + \dots + m_n \vec{R}_n$$

$$M_{total} \frac{d\vec{R}_{cm}}{dt} = m_1 \frac{d\vec{R}_1}{dt} + m_2 \frac{d\vec{R}_2}{dt} + \dots + m_n \frac{d\vec{R}_n}{dt}$$

We recognize the derivatives on the right side as the individual velocities of each of the particles in the collection. On the left side, we introduce the velocity of the center of mass.

$$M_{total} \vec{v}_{cm} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_n \vec{v}_n$$

Again, we differentiate the above with respect to time.

$$M_{total} \vec{v}_{cm} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_n \vec{v}_n$$

$$M_{total} \frac{d\vec{v}_{cm}}{dt} = m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + \dots + m_n \frac{d\vec{v}_n}{dt}$$

and again, we recognize the derivatives as accelerations.

$$M_{total} \vec{a}_{cm} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 + \dots + m_n \vec{a}_n$$

We are also in a position to make an interesting discovery. Examining the above equation, each  $m_i \vec{a}_i$  on the right side represents a force on that particle.

$$M_{total} \vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_N$$

Now consider the situation that particle 2 and particle 3 are involved in a force interaction, then by the 3<sup>rd</sup> law of Newton, these two forces would be equal and opposite,  $F_2 = -F_3$ . These two forces would then add to zero. All forces internal to the collection of particles will add to zero and that leaves only the forces external to the collection of particles.

$$M_{total} \vec{a}_{cm} = \sum \vec{F}_{External}$$

The sum of the forces, external to the collection of particles, will cause the center of mass to accelerate. We are now in a position to summarize the properties of the center of mass.

**4. Properties.** The center of mass has the following properties:

1. The center of mass has a coordinate position. The position is given by  $M_{total} \vec{R}_{cm} = m_1 \vec{R}_1 + m_2 \vec{R}_2 + m_3 \vec{R}_3 + \dots + m_n \vec{R}_n$
2. The center of mass may have a velocity. The velocity is given by  $M_{total} \vec{v}_{cm} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_n \vec{v}_n$
3. The center of mass may have an acceleration. The acceleration is given by  $M_{total} \vec{a}_{cm} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 + \dots + m_n \vec{a}_n$

4. The center of mass has a mass equal to the total mass of all the particles in the collection which define it. Thus, the center of mass will resist being accelerated.
5. The center of mass will only respond to an external force, that is, a force of external origin to the collection.  $M_{total} \vec{a}_{cm} = \sum \vec{F}_{External}$
6. The center of mass may have a momentum as indicated in #2 above. It will respond to an external Impulse.
7. The center of mass coordinate is a representative location for the placement of the weight vector,  $M_{total}g$ , for the collection of particles.
8. The center of mass can change Gravitational Potential Energy.
9. The center of mass can change Kinetic Energy.
10. Work may be done on or by the center of mass however this work must be as a result of an external force .

There are some properties which can not be applied to the center of mass. They are:

- The center of mass does not represent the resistance to angular acceleration. The center of mass can not rotate...it is a point.
- The center of mass can not under go plastic (permanent) deformation...it is a point. Thus, deformation energies for the center of mass do not exist.
- The center of mass can not contain any Thermal quantities. It can not change temperature, or even have a temperature. Thermal quantities are in fact, statistical quantities. As such, many particles are required to give meaning to the thermal quantities.

The above concepts can only apply to collections of multiple particles, not a single center of mass particle.

## The Flow $F = ma$ .

**1. Development.** A second adaptation of Newton's second law involves the application of the product rule in calculus. Starting with the 2<sup>nd</sup> law,

$$\sum \vec{F}_{ex} = \frac{d(m\vec{v})}{dt}$$

we apply the product rule:

$$\sum \vec{F}_{ex} = m(t) \frac{d\vec{v}}{dt} + \vec{v}_r \frac{dm}{dt}$$

In this adaptation, the mass of the system can be a function of time. This implies that the system is not bound to mass conservation but rather that mass may enter and leave the system at will. In contrast, using  $F = ma$ , the systems mass must be conserved. The expanded version of the second law above acquires the prefix Flow  $F=ma$  due to its ability to account for mass flow across the system boundary.

The Flow  $F = ma$  equation is often used in practical engineering and design. Consider a system that is itself accelerating. Also consider that this system is accepting and rejecting multiple streams of matter, each stream with a different velocity, situations of this complexity are usually addressed with some variant of the Flow  $F = ma$ .

The variable, " $v_r$ " in the above equation is the velocity of the system, measured from the frame of reference of the only other meaningful entity of concern, the matter flowing into (or out of) the system; hence the subscript "r" standing for "relative".

An organizational approach to the application of the Flow  $F=ma$  equation for problems involving multiple streams of matter entering and leaving the system is to expand the flow term vector,  $\vec{v}_r \frac{dm}{dt}$ , into a series of vector components which add to make it; one component for each physical flow stream. In doing so, the Flow  $F=ma$  takes on its final form.

The Flow  $F = ma$  is:

$$\sum \vec{F}_{ex} = m(t) \frac{d\vec{v}}{dt} + \sum_{i=1}^n \left( \vec{v}_r \frac{dm}{dt} \right)_i$$

The “system” in this application is called a Control Volume. A region in space that contains mass. It is for all purposes an accounting boundary.

$\sum \vec{F}_{ex}$ , is the sum of the external force vectors acting on the system.

$m(t)$ , is the instantaneous mass of the system (a function of time).

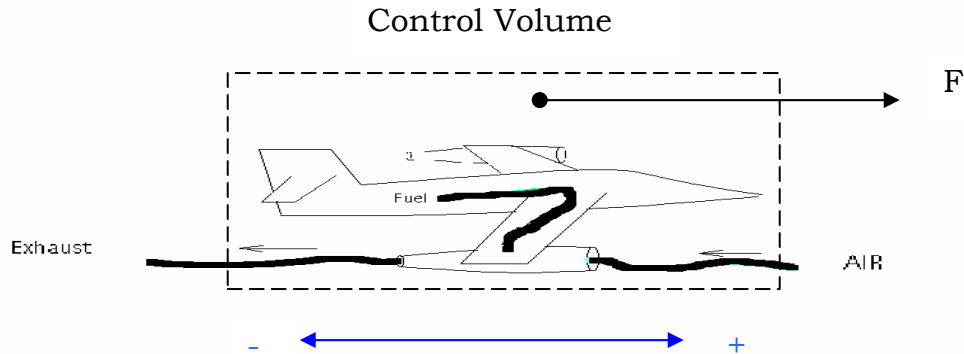
$dv/dt$ , is the acceleration of the system.

(the above definitions are similar to those used in  $F=ma$ )

$dm/dt$ , is the scalar rate at which the system gains mass. It is positive for mass gain, and negative for mass leaving the system. Note that this sign convention is consistent with the system mass,  $m(t)$ . The sign convention is understood in an accounting sense; Not a vector sense.

$V_r$ , is a vector. It is the velocity of the system, measured from the frame of reference of its associated flow stream. The sense of this value is determined by putting yourself in the flow stream, coordinate system in your lap...your velocity is of course zero. Now assess the direction and speed of the system based on the vector coordinate system. ( $V_r$  is not the velocity of the flow stream although some branches of engineering do further adapt the equation for that perspective. We will not.)

## 2. An Example Application.



### Problem:

The jet engines of an airplane flying at a constant speed of 550 ft/s in still air use fuel at the rate of .20 slugs/s. The fuel is combined with 4.80 slugs of air each second. The mixture is burned and ejected rearward out a nozzle with a speed of 1700 f/s relative to the engine. What is the net horizontal force on the airplane?

### Solution:

We assign a vector coordinate system, positive points right. The dimension is horizontal. The control volume is the entire airplane by choice. There are two flow streams involved: Air in, and Exhaust out. The fuel flow is internal to the control volume. In the flow  $F=ma$ ,  $dv/dt$  (the acceleration of the control volume) is zero since the airplane has constant velocity. The “ $m(t)a$ ” term drops out. This leaves,

$$F_{ex} = v_r \frac{dm}{dt}_{Air} + v_r \frac{dm}{dt}_{Exhaust}$$

We assume that in steady flight, mass into the engine equals mass out of the engine. This gives an exhaust mass flow rate of  $-5.0$  slugs/sec. Sitting in the still air, we see the airplane moving to the right...POSITIVE...and closing on us at 550 f/s.  $V_r$  for the air is  $+550$  ft/s.

Sitting in the exhaust stream, we see the airplane moving to the right...POSITIVE...and leaving us at 1700 ft/s.  $V_r$  for the exhaust is  $+1700$  ft/s.

Substituting these values in the equation leads to,

$$F_{ex} = (+ 550 \text{ ft/s})(+ 4.8 \text{ slugs/s}) + (+ 1700 \text{ ft/s})(- 5.0 \text{ slug/s})$$

$$F_{ex} = -5860 \text{ lbs}$$

Interpretation of this result is that the net force on the airplane is to the left at a value of 5860 lbs. This force is called the aerodynamic drag. The aerodynamic drag force is due to a complex normal and shear interaction between the airplane and the air around it.

It seems unsettling to the unfamiliar that it is possible to have a net force while also having zero acceleration. In the context of the Flow  $F=ma$ , it is quite possible. In the Flow context, there is no external force called “thrust”, although some branches of engineering do invent the concept and assign it to the absolute value of the flow resultant. They would say that the thrust in this problem is 5860 lbs, and in the non-flow way of thinking, the thrust would equal the drag.